Exponential Sums Related to Maass Forms

Jesse Jääsaari[*] and Esa V. Vesalainen[†]

Abstract

We estimate short exponential sums weighted by the Fourier coefficients of a Maass form and discuss how the results depend on the growth of the Fourier coefficients in question. As a byproduct of these considerations, we can slightly extend the range of validity of a short exponential sum estimate for holomorphic cusp forms. The short estimates allow us to reduce smoothing errors. In particular, we prove an analogue of an approximate functional equation previously proven for holomorphic cusp form coefficients.

As an application, we remove the logarithm from the classical upper bound for long linear sums weighted by Fourier coefficients of Maass forms, the resulting estimate being the best possible. This also involves improving the upper bounds for long linear sums with rational additive twists, the gains again allowed by the estimates for the short sums. Finally, we shall use the approximate functional equation to bound somewhat longer short exponential sums.

1 Introduction and the main results

1.1 Maass forms

Let ψ be a Maass form for the full modular group, corresponding to an eigenvalue $1/4+\kappa^2$ of the hyperbolic Laplacian, and with the Fourier expansion

$$\psi(x+yi)=y^{1/2}\sum_{n\neq 0}t(n)\,K_{i\kappa}(2\pi\,|n|\,y)\,e(nx)\,,$$

where $x\in\mathbb{R}$ and $y\in\mathbb{R}_+$. We may assume without loss of generality that ψ is even or odd, i.e. that $t(-n)=t(n)$ for all $n\in\mathbb{Z}_+$, or that $t(-n)=-t(n)$ for all $n\in\mathbb{Z}_+$. For standard references on Maass forms we refer to [Iw, Mo]. It is perhaps wise to remark already here that all the implicit constants in this article are allowed to depend on ψ.

The Fourier coefficients $t(n)$ satisfy a bound of the kind

$$t(n)\ll n^{\vartheta+\varepsilon}$$

for some $\vartheta\in[0,\infty[$. Arguably the simplest admissible exponent is $\vartheta=1/2$. However, our results below require $\vartheta<1/6$ to be interesting. Fortunately, the best known exponent is $\vartheta=7/64$, due to Kim and Sarnak [KS]. The Ramanujan–Petersson conjecture for Maass forms declares that $\vartheta=0$ is admissible. On average, the Fourier coefficients are of constant size. In particular, we have a Rankin–Selberg type estimate for the Fourier coefficients. One such result is the following (see e.g. [Iw], Chapter 8):

$$\int_0^1\left|\sum_{n\leqslant M}t(n)\,e(n\alpha)\right|^2\mathrm{d}\alpha=\sum_{n\leqslant M}|t(n)|^2=A\,M+O(M^{7/8}),\tag{1}$$

where A is a positive real constant depending on ψ.

[*]University of Helsinki, Department of Mathematics and Statistics, P.O.Box 68 (Gustaf Hällströmin katu 2b), FI-00014 University of Helsinki, FINLAND

[†]Mathematics and Statistics, Åbo Akademi University, Domkyrkotorget 1, 20500 Åbo, FINLAND

1.2 Objects of study and motivation

In the following we will consider linear exponential sums of the form

$$\sum_{M \leqslant n \leqslant M+\Delta} t(n)\, e(n\alpha),$$

where $M \in [1, \infty[$, $\Delta \in [1, M]$ and $\alpha \in \mathbb{R}$. When $\Delta = o(M)$, we call such sums short.

The reasons for considering such sums are manifold. First of all, the Fourier coefficients $t(n)$ are interesting mathematical objects which are not as well understood as one might wish. As trigonometric polynomials, the exponential sums above contain all the information about the Fourier coefficients and thus provide an interesting window into their behaviour. The study of Maass forms and their L-functions also naturally leads to exponential sums weighted by the corresponding Fourier coefficients of which the above linear sums are an important special case.

When α is a rational number h/k, the problem of estimating long sums with $\Delta = M$ is very analogous to classical problems in analytic number theory, such as the problems of estimating the error terms in the circle and Dirichlet divisor problems. Furthermore, the problem of estimating such sums with $\Delta = o(M)$ provides an analogue for problems such as studying the behaviour of the afore-mentioned error terms in short intervals. For further information about these classical topics, see e.g. Chapter 13 of [Iv1] or [T].

Finally, good estimates for the short exponential sums above can sometimes be used to reduce smoothing error. An example of such an application is given e.g. by Theorems 4 and 6 below.

For holomorphic cusp forms, short exponential sums have been studied by Jutila [Ju5], and the best known bounds are due to Ernvall-Hytönen and Karppinen [EK, E1].

It is interesting to study how sensitive the arguments used for holomorphic cusp forms are to the value of ϑ. In a sense, the strictly positive value of ϑ is the main difference between the holomorphic and non-holomorphic cases: in both cases one applies heavily the corresponding Voronoi summation formula, and even though the Voronoi summation formulae have a different appearance, what remains after the Bessel functions have been cashed in in terms of their asymptotics is similar.

1.3 The results: Bounds for short exponential sums with applications

The following is a Maass form analogue of the related estimate for holomorphic cusp forms due to Ernvall-Hytönen and Karppinen, Theorem 5.5 in [EK]. The proof is based on techniques analogous to those in [EK].

Theorem 1. *Let $M \in [1, \infty[$ and let $\Delta \in [1, M]$ be such that $\Delta \ll M^{2/3}$. Then*

$$\sum_{M \leqslant n \leqslant M+\Delta} t(n)\, e(n\alpha) \ll \Delta^{1/6-\vartheta}\, M^{1/3+\vartheta+\varepsilon},$$

uniformly for $\alpha \in \mathbb{R}$. This is better than estimating via absolute values when we have $M^{2/(5+6\vartheta)} \ll \Delta \ll M^{2/3}$.

When $\Delta = M^{2/3}$ this gives the upper bound $\ll M^{\vartheta/3+4/9+\varepsilon}$, and so splitting a longer sum into sums of this length and estimating the subsums separately gives the following bound for longer sums.

Corollary 2. *Let $M \in [1, \infty[$ and let $\Delta \in [1, M]$ be such that $M^{2/3} \ll \Delta \ll M$. Then*

$$\sum_{M \leqslant n \leqslant M+\Delta} t(n)\, e(n\alpha) \ll \Delta\, M^{\vartheta/3-2/9+\varepsilon}.$$

This is better than the bound $\ll M^{1/2+\varepsilon}$ when $M^{2/3} \ll \Delta \ll M^{(13-6\vartheta)/18}$.

Actually, Theorem 1 is valid for a slightly larger range of Δ than Theorem 5.5 in [EK] is. In fact, with a minor modification [E7], the proof of Theorem 5.5 of [EK] can be easily modified to give the analogous result for holomorphic cusp forms:

Theorem 3. *Let us consider a fixed holomorphic cusp form of weight $\kappa \in \mathbb{Z}_+$ for the full modular group with the Fourier expansion*

$$\sum_{n=1}^{\infty} a(n)\, n^{(\kappa-1)/2}\, e(nz)$$

for $z \in \mathbb{C}$ with $\Im z > 0$. Also, let $M \in [1,\infty[$, $\Delta \in [1, M]$, and let $\alpha \in \mathbb{R}$. If $\Delta \ll M^{2/3}$, then

$$\sum_{M \leqslant n \leqslant M+\Delta} a(n)\, e(n\alpha) \ll \Delta^{1/6}\, M^{1/3+\varepsilon},$$

where the implicit constant depends only on the underlying cusp forms and ε. Similarly, if $M^{2/3} \ll \Delta$, then

$$\sum_{M \leqslant n \leqslant M+\Delta} a(n)\, e(n\alpha) \ll \Delta\, M^{-2/9+\varepsilon}.$$

The proof of Theorem 1 depends on an estimate for short non-linear sums, analogous to Theorem 4.1 in [EK]. Fortunately, the proof in [EK] works almost verbatim for Maass forms and we shall indicate the differences later. On the other hand, the proof of the non-linear estimate requires a transformation formula of a certain shape for smoothed exponential sums, and this particular result does not seem to have been worked out before yet. Thus, in Section 4, we will give an analogue of the relevant Theorem 3.4 of Jutila's monograph [Ju4], which considers smooth sums with holomorphic cusp form coefficients, indicating the crucial differences to the proof of Theorem 3.4 of [Ju4]. An analogue of Theorem 3.2 of [Ju4] has been given by Meurman in [Me1].

The following estimates provide a concrete example of how estimates for short sums allow one to reduce smoothing errors thereby leading to improved upper bounds.

Theorem 4. *Let $M \in [1,\infty[$, $h \in \mathbb{Z}$, $k \in \mathbb{Z}_+$ and $(h,k) = 1$. Also, let $\delta \in\,]0, 1/2[$ and assume that $k \ll M^{1/2-\delta}$. Then*

$$\sum_{n \leqslant M} t(n)\, e\!\left(\frac{nh}{k}\right) \ll_\delta k^{2/3}\, M^{1/3+\vartheta/3+\varepsilon}.$$

When $M^{3/(5+6\vartheta)-1/2+\vartheta} \ll k \ll M^{5/18+\vartheta/3}$, we have

$$\sum_{n \leqslant M} t(n)\, e\!\left(\frac{nh}{k}\right) \ll k^{(1-6\vartheta)/(4-6\vartheta)}\, M^{3/(8-12\vartheta)+\varepsilon}.$$

Similarly, for $M^{5/18+\vartheta/3} \ll k \ll M^{1/2-\delta}$, we have

$$\sum_{n \leqslant M} t(n)\, e\!\left(\frac{nh}{k}\right) \ll_\delta k^{2/3}\, M^{7/27+\vartheta/9+\varepsilon}.$$

The case $k = 1$ was considered by Hafner and Ivić [HI] who essentially obtained the bound $\ll M^{1/3+\vartheta/3}$. For $k = 1$, the best result to date seems to be $\ll M^{1027/2827+\varepsilon}$, due to Lü [Lü]. The twisted upper bound $k^{2/3}\, M^{1/3+\vartheta/3+\varepsilon}$ was first obtained by Meurman [Me2]. Similar reduction for certain ranges of k in the case of holomorphic cusp forms have recently been proved in [V]. The proof is analogous to the approach of [Iv2].

It is of interest to note here that for small enough k, the rationally twisted sum has on average (in the mean square sense) the order of magnitude $k^{1/2}\,M^{1/4}$. This kind of result was first proven by Cramér [C] for the error term in the Dirichlet divisor problem. Jutila [Ju3] extended this to the divisor problem with rational additive twists, and in [Ju4] Jutila proved the analogous result for holomorphic cusp forms. We shall elaborate on this in the last section.

Theorem 3 allows us to improve Theorem 1 from [V] in the range $k \gg M^{1/4}$:

Corollary 5. *Let $a(n)$ be the Fourier coefficients of a holomorphic cusp form as in Theorem 3. Then, for coprime integers h and k with $M^{1/10} \ll k \ll M^{5/18}$, we have*

$$\sum_{n\leqslant M} a(n)\, e\left(\frac{nh}{k}\right) \ll k^{1/4}\, M^{3/8+\varepsilon},$$

and for $M^{5/18} \ll k \ll M^{1/2-\varepsilon}$, we have

$$\sum_{n\leqslant M} a(n)\, e\left(\frac{nh}{k}\right) \ll k^{2/3}\, M^{7/27+\varepsilon}.$$

1.4 The results: an approximate functional equation and applications

Wilton [W2] proved an approximate functional equation for exponential sums involving the divisor function. Jutila [Ju3] extended this to sums with additive twists, and in [Ju5] he proved an analogue for holomorphic cusp forms. In [E1] Ernvall-Hytönen improved the error term. The following is an analogue of Ernvall-Hytönen's result, and the proof is analogous to that in [E1]. We write $\overline{h}$ for an integer such that $h\overline{h} \equiv 1 \pmod{k}$. Also, to simplify the notation, we write

$$T(M,\Delta;\alpha) = \sum_{M\leqslant n\leqslant M+\Delta} t(n)\, e(n\alpha).$$

Theorem 6. *Let $\alpha \in \mathbb{R}$ have the rational approximation $\alpha = \frac{h}{k} + \eta$, where h and k are coprime integers with $1 \leqslant k \leqslant M^{1/4}$ and $|\eta| \leqslant k^{-1}\,M^{-1/4}$. Furthermore, let $M \in [1,\infty[$ and $\Delta \in [1, M]$. If $k^2\,\eta^2\,M \gg 1$, then*

$$\frac{T(M,\Delta;\alpha)}{M^{1/2}} = \frac{T(k^2\,\eta^2\,M, k^2\,\eta^2\,\Delta;\beta)}{(k^2\,\eta^2\,M)^{1/2}} + O\big((k^2\,\eta^2\,M)^{\vartheta/2-1/12+\varepsilon}\big),$$

where $\beta = -\dfrac{\overline{h}}{k} - \dfrac{1}{k^2\eta}$.

Crudely speaking, the terms in the sum on the right-hand side arise essentially from integrals with stationary points coming from the application of the Voronoi type summation formula to a suitably smoothed version of the left-hand side.

Wilton [W1] proved that for the normalized Fourier coefficients $a(n)$ of a fixed holomorphic cusp form,

$$\sum_{n\leqslant M} a(n)\, e(n\alpha) \ll M^{1/2}\, \log M,$$

uniformly in $\alpha \in \mathbb{R}$. The Rankin–Selberg bound on the mean square of Fourier coefficients implies that

$$\int_0^1 \left|\sum_{n\leqslant M} a(n)\, e(n\alpha)\right|^2 \mathrm{d}\alpha = \sum_{n\leqslant M} |a(n)|^2 = A\,M + O(M^{3/5}),$$

for a certain positive real constant A depending on the underlying cusp form, and so at most the logarithm can be removed from Wilton's estimate, and this indeed was done by Jutila [Ju5]. For Maass forms, the estimate analogous to Wilton's was proved by Epstein, Hafner and Sarnak [EHS, H]. The following estimate is an analogue of Jutila's logarithm removal, and its proof is largely analogous to the arguments in [Ju5].

Theorem 7. *We have*

$$\sum_{n\leqslant M} t(n)\, e(n\alpha) \ll M^{1/2},$$

uniformly in $\alpha\in\mathbb{R}$.

This is sharp in view of (1).

With the approximate functional equation at hand, we may prove further estimates for short sums.

Theorem 8. *Let* $M\in[1,\infty[$ *and* $\Delta\in[1,M]$ *with* $M^{2/3}\ll\Delta\ll M^{3/4}$, *and let* $\alpha\in\mathbb{R}$. *Then*

$$\sum_{M\leqslant n\leqslant M+\Delta} t(n)\, e(n\alpha) \ll M^{3/8+(3+12\vartheta)/(32+48\vartheta)+\varepsilon} + \Delta\, M^{-1/4+3\vartheta/(32+48\vartheta)+\varepsilon}.$$

In particular, for $\vartheta=7/64$ *we have*

$$\sum_{M\leqslant n\leqslant M+\Delta} t(n)\, e(n\alpha) \ll M^{585/1192+\varepsilon} + \Delta\, M^{-575/2384+\varepsilon},$$

which is better than the estimate $\ll M^{1/2+\varepsilon}$ *for* $\Delta\ll M^{1767/2384}$, *and for* $\vartheta=0$, *we have*

$$\sum_{M\leqslant n\leqslant M+\Delta} t(n)\, e(n\alpha) \ll M^{15/32+\varepsilon} + \Delta\, M^{\varepsilon-1/4}.$$

1.5 Complementary Ω-results

Finally, it is naturally interesting to consider what are the limits of estimating short sums. In [E3] Ernvall-Hytönen proved that, if $d\in\mathbb{Z}_+$ is a fixed integer such that $t(d)\neq 0$, then

$$\sum_{M\leqslant n\leqslant M+\Delta} t(n)\, e(n\alpha)\, w(n) \asymp \Delta\, M^{-1/4},$$

where w is a suitable weight function, $\alpha=\sqrt{d}/\sqrt{M}$, and $M^{1/2+\varepsilon}\ll\Delta\leqslant d^{-1/2}\, M^{3/4}$. This immediately implies that for this range of lengths Δ,

$$\sum_{M\leqslant n\leqslant M+\Delta} t(n)\, e\!\left(\frac{n\sqrt{d}}{\sqrt{M}}\right) = \Omega(\Delta\, M^{-1/4}). \tag{2}$$

This result also has counterparts for the divisor function and Fourier coefficients of holomorphic cusp forms in the papers of Ernvall-Hytönen and Karppinen [EK] and Ernvall-Hytönen [E1, E2]. The above Ω-result implies that the bound

$$\sum_{M\leqslant n\leqslant M+\Delta} t(n)\, e(n\alpha) \ll M^{1/2}$$

from Theorem 7 is sharp for $M^{3/4}\ll\Delta\ll M$.

For sums of length $\Delta\ll M^{1/2}$, it turns out that square root cancellation is the best that could be hoped for. Essentially, combining the truncated Voronoi identity of Meurman [Me2] with the arguments of Jutila [Ju2], one gets the following mean square asymptotics

$$\int_M^{2M} \Bigg| \sum_{x\leqslant n\leqslant x+\Delta} t(n) \Bigg|^2 \, \mathrm{d}x \asymp \Delta\, M,$$

for $M^{2\vartheta+\varepsilon} \ll \Delta \ll M^{1/2-\varepsilon}$. In fact, a sharper result could be obtained, but this is enough for the relevant Ω-result. The paper [Ju2] actually considered the behaviour of the error terms in the Dirichlet divisor problem and the second moment for the Riemann ζ-function in short intervals, but the proof for the divisor function carries through fairly easily for Fourier coefficients of holomorphic cusp forms or Maass forms. In the last section, we will discuss the second moments with more details, and add here only that for holomorphic cusp forms second moments of rationally additively twisted short sums have been considered in the works [E5, E6, V].

We would like to emphasize that there are reasons to believe that even if the best possible upper bounds conform to the above Ω-results, they are likely to be very difficult to prove. For example, the conjectural upper bounds

$$\sum_{M\leqslant n\leqslant M+M^{1/2}} a(n)\, e(n\alpha) \ll M^{1/4+\varepsilon} \quad \text{and} \quad \sum_{M\leqslant n\leqslant M+M^{1/2}} t(n)\, e(n\alpha) \ll M^{1/4+\varepsilon}$$

would be analogous to the conjectural upper bound

$$\Delta(M+M^{1/2}) - \Delta(M) \ll M^{1/4+\varepsilon}$$

for the error term in the Dirichlet divisor problem. Jutila [Ju1] has proved that if the last estimate is true, then Riemann's zeta-function satisfies the bound $\zeta(1/2+it) \ll t^{3/20+\varepsilon}$ on the critical line, and this exponent $3/20$ is better than the best known exponent $53/342$ due to Bourgain [B] or Huxley's exponent $32/205$ [H2].

1.6 Further complementary results: uniformity and higher rank

We would like to say a few words about Maass forms for $\mathrm{GL}(n)$. For them, a Voronoi summation formula exists and was implemented in [MS1, MS2, GL1, GL2]. It has been applied to exponential sums weighted by Fourier coefficients of $\mathrm{GL}(n)$ Maass forms. As examples, we mention the works [Mi, LY, E4, EJV, G]. In particular, [Mi] gives an upper bound for long linear sums in $\mathrm{GL}(3)$, and [E4, EJV] give higher rank analogues of the above mentioned Ω-result (2).

We have only considered a fixed cusp form for the full modular group. The dependence of the upper bound for long linear sums on the underlying cusp form has been considered in [LY] and [G]. The discussion of the Farey and similar methods for holomorphic cusp forms in [H1] also considers the depence on the underlying cusp forms, and the papers [Me1] and [Me2] consider the dependence on the underlying Maass form.

1.7 Notation

All the implicit constants are allowed to depend on the underlying Maass form, and ε, which denotes an arbitrarily small fixed positive number, which is not necessarily the same on each occurrence. Implicit constants depend also on chosen positive integers J and K, when they appear.

The symbols $\ll$, $\gg$, $\asymp$, and O are used for the usual asymptotic notation: for complex valued functions f and g in some set Ω, the notation $f \ll g$ means that $|f(x)| \leqslant C\,|g(x)|$ for all $x \in \Omega$ for some implicit constant $C \in \mathbb{R}_+$. When the implicit constant depends on some parameters $\alpha, \beta, \ldots$, we use $\ll_{\alpha,\beta,\ldots}$ instead of mere $\ll$. The notation $g \gg f$ means $f \ll g$, and $f \asymp g$ means $f \ll g \ll f$.

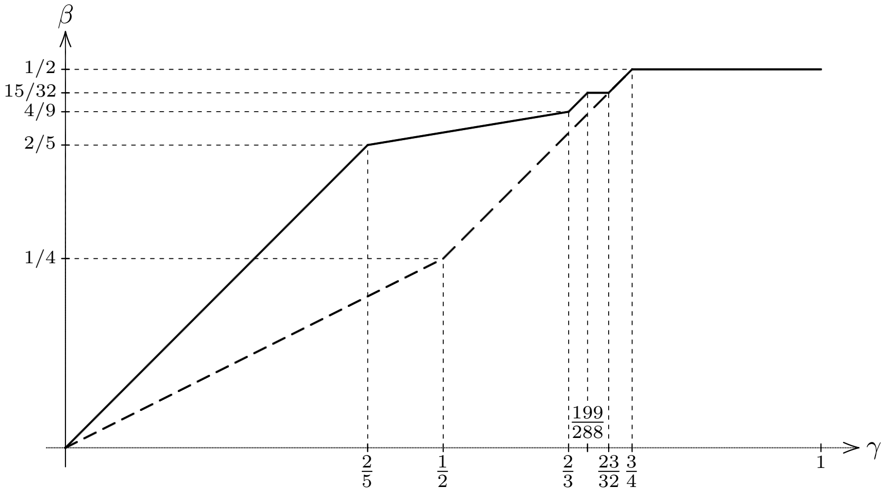

Figure 1. Estimates for short linear sums related to holomorphic cusp forms: a point $\langle \gamma, \beta \rangle$ on the solid thick line or on the dotted thick line signifies an estimate of the form

$$\sum_{M \leqslant n \leqslant M+M^\gamma} a(n)\, e(n\alpha) \ll M^{\beta+\varepsilon}, \quad \text{or} \quad \sum_{M \leqslant n \leqslant M+M^\gamma} a(n)\, e(n\alpha) = \Omega(M^\beta),$$

respectively. The first estimate holds uniformly in $\alpha \in \mathbb{R}$. The first upper bound segment comes from estimating by absolute values with Deligne's estimate for individual Fourier coefficients from [D], the second and third segments from Theorem 5.5 of [EK] and Theorem 3, the fourth and fifth segments from Theorem 5.16 in [EK], and the horizontal sixth segment from Wilton's and Jutila's estimates [W1, Ju5]. The first lower bound segment follows from the work of Jutila [Ju2], and the second follows from Theorem 6.1 in [EK]. The upper bounds in the range $M^{23/32+\varepsilon} \ll \Delta \ll M$ are sharp.

The symbol $\sum'_{a \leqslant n \leqslant b}$ signifies summation over the integers n with $a \leqslant n \leqslant b$, with possible terms corresponding to a and b halved if a or b is an integer. The symbol $\sum_{L \leqslant X, \text{dyadic}}$ signifies summation over the values $L = X$, $L = X/2$, $L = X/4$, …

Finally, the characteristic function of a set B is denoted by χ_B, and $e(x)$ denotes $e^{2\pi i x}$ for all $x \in \mathbb{R}$.

2 The Voronoi type summation formula for Maass forms

The main tool in the following is a Voronoi type summation formula for Maass forms with rational additive twists, proved by Meurman [Me2]. The following result is Theorem 2 in [Me2].

Theorem 9. *For a function $f \in C^1([a,b])$, where $a < b$ are positive real numbers, and for a positive integer k and an integer h coprime to k, we have*

$$\sum_{a \leqslant n \leqslant b}{}' t(n)\, e\left(\frac{nh}{k}\right) f(n)$$
$$= \frac{\pi\, i}{k \sinh \pi\kappa} \sum_{n=1}^{\infty} t(n)\, e\left(\frac{-n\overline{h}}{k}\right)$$

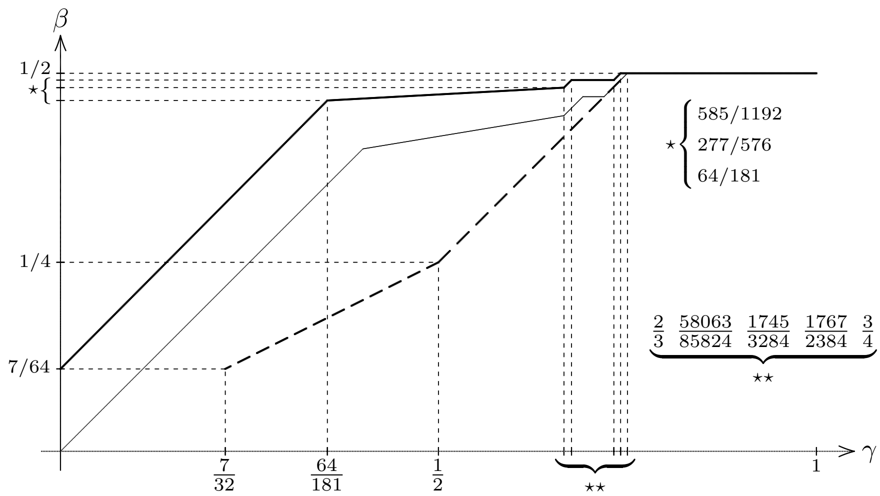

Figure 2. Estimates for short linear sums related to Maass forms: a point $\langle \gamma, \beta \rangle$ on the solid thick line or on the dotted thick line signifies an estimate of the form

$$\sum_{M \leqslant n \leqslant M+M^\gamma} a(n)\, e(n\alpha) \ll M^{\beta+\varepsilon}, \quad \text{or} \quad \sum_{M \leqslant n \leqslant M+M^\gamma} a(n)\, e(n\alpha) = \Omega(M^\beta),$$

respectively, when ϑ is taken to be $7/64$. The first estimate holds uniformly in $\alpha \in \mathbb{R}$. The first upper bound segment comes from estimating by absolute values, the second and third from Theorem 1, the fourth and fifth segments from Theorem 8, and the sixth horizontal segment from [EHS, H] and Theorem 7. The first lower bound segment follows from the arguments in [Ju2] (but see also Theorem 35 below), the second lower bound segment comes from Theorem 2 in [E3]. The upper bounds in the range $M^{3/4} \ll \Delta \ll M$ are sharp. If ϑ can be taken to be zero, then this picture reduces to the one in Figure 1.

$$\cdot \int_a^b \left(J_{2i\kappa}\left(\frac{4\pi\sqrt{nx}}{k}\right) - J_{-2i\kappa}\left(\frac{4\pi\sqrt{nx}}{k}\right) \right) f(x)\,\mathrm{d}x$$
$$+ \frac{4\cosh \pi\kappa}{k} \sum_{n=1}^{\infty} t(-n)\, e\left(\frac{n\overline{h}}{k}\right) \int_a^b K_{2i\kappa}\left(\frac{4\pi\sqrt{nx}}{k}\right) f(x)\,\mathrm{d}x.$$

The following upper bound for the K-Bessel function will be enough for estimating all the integrals involving it:

$$K_\nu(x) \ll_\nu x^{-1/2}\, e^{-x} \ll_A x^{-A},$$

where $A > 0$ is fixed and $x \gg 1$. This follows from (5.11.9) in [Le]. Here ν is fixed. In particular, we may estimate

$$K_{2i\kappa}\left(\frac{4\pi\sqrt{nx}}{k}\right) \ll_A k^A\, n^{-A/2}\, x^{-A/2}, \tag{3}$$

for $x \in [1, \infty[$ and $n, k \in \mathbb{Z}_+$ satisfying $nx \gg k^2$.

For the J-Bessel functions appearing in the Voronoi summation formula, we have, for every $K \in \mathbb{Z}_+$, the asymptotics

$$J_{2i\kappa}\left(\frac{4\pi\sqrt{nx}}{k}\right) - J_{-2i\kappa}\left(\frac{4\pi\sqrt{nx}}{k}\right) = \frac{k^{1/2}\,\sinh\pi\kappa}{\pi\sqrt{2}}\, n^{-1/4}\, x^{-1/4}$$
$$\cdot \sum_{\pm}(\pm 1)\, e\left(\mp\frac{1}{8} \pm \frac{2\sqrt{nx}}{k}\right)\left(1 + \sum_{\ell=1}^{K} c_\ell^{\pm}\, k^\ell\, n^{-\ell/2}\, x^{-\ell/2}\right)$$
$$+ O_K\left(k^{1/2+(K+1)}\, n^{-1/4-(K+1)/2}\, x^{-1/4-(K+1)/2}\right), \tag{4}$$

again for $n\,x \gg k^2$. This follows from (5.11.6) of [Le].

3 Theorems on exponential integrals

The use of the Voronoi summation formula leads to many exponential integrals, and so we will introduce several facts about such integrals.

Let us consider an interval $[M_1, M_2] \subseteq \mathbb{R}_+$, and let $U \in \mathbb{R}_+$ and $J \in \mathbb{Z}_+$ be such that $2JU < M_2 - M_1$. Following [Ju4], we introduce weight function η_J by requiring that

$$\int\limits_{M_1}^{M_2} \eta_J(x)\, h(x)\, \mathrm{d}x = U^{-J} \int\limits_0^U \int\limits_0^U \cdots \int\limits_0^U \int\limits_{M_1+u_1+\ldots+u_J}^{M_2-u_1-\ldots-u_J} h(x)\, \mathrm{d}x\, \mathrm{d}u_J \cdots \mathrm{d}u_2\, \mathrm{d}u_1 \tag{5}$$

for any integrable function h on $\mathbb{R}$. It is not too difficult to see that actually η_J is given by the convolution

$$\eta_J = \frac{1}{U}\chi_{[0,U]} * \frac{1}{U}\chi_{[0,U]} * \cdots * \frac{1}{U}\chi_{[0,U]} * \chi_{[M_1, M_2-JU]},$$

with $U^{-1}\chi_{[0,U]}$ appearing J times. In particular, η_J is $J-1$ times continuously differentiable on $\mathbb{R}$, and supported in $[M_1, M_2]$.

The following saddle point theorem is a special case of Theorem 2.2 in [Ju4].

Theorem 10. *Let us consider an interval $[M_1, M_2] \subseteq \mathbb{R}_+$, let $\mu \in \mathbb{R}_+$, and let D stand for the domain*

$$D = \big\{z \in \mathbb{C} : |z - x| < \mu \text{ for some } x \in [M_1, M_2]\big\}.$$

Let $f, g\colon D \longrightarrow \mathbb{C}$ be holomorphic, let $F, G \in \mathbb{R}_+$, and assume that $F \gg 1$, $f(x) \in \mathbb{R}$, $f''(x) > 0$ and $f''(x) \gg F\,\mu^{-2}$ for $x \in [M_1, M_2]$, and that $f'(z) \ll F\,\mu^{-1}$ and $g(z) \ll G$ for $z \in D$.

Next, let $U \in \mathbb{R}_+$ and $J \in \mathbb{Z}_+$ be such that $2\,J\,U < M_2 - M_1$, $U \gg \mu\,F^{-1/2}$, and let η_J denote the weight function defined as above, namely the convolution

$$\eta_J = \frac{1}{U}\chi_{[0,U]} * \frac{1}{U}\chi_{[0,U]} * \cdots * \frac{1}{U}\chi_{[0,U]} * \chi_{[M_1, M_2-JU]},$$

with $U^{-1}\chi_{[0,U]}$ appearing J times.

Finally, let $\alpha \in \mathbb{R}$, and let $x_0 \in\,]M_1, M_2[$ be such that $f'(x_0) + \alpha = 0$. Then

$$\int\limits_{M_1}^{M_2} g(x)\, e(f(x) + \alpha x)\, \eta_J(x)\, \mathrm{d}x$$
$$= \xi_J(x_0)\, g(x_0)\, f''(x_0)^{-1/2}\, e\big(f(x_0) + \alpha x_0 + 1/8\big) + \textit{error},$$

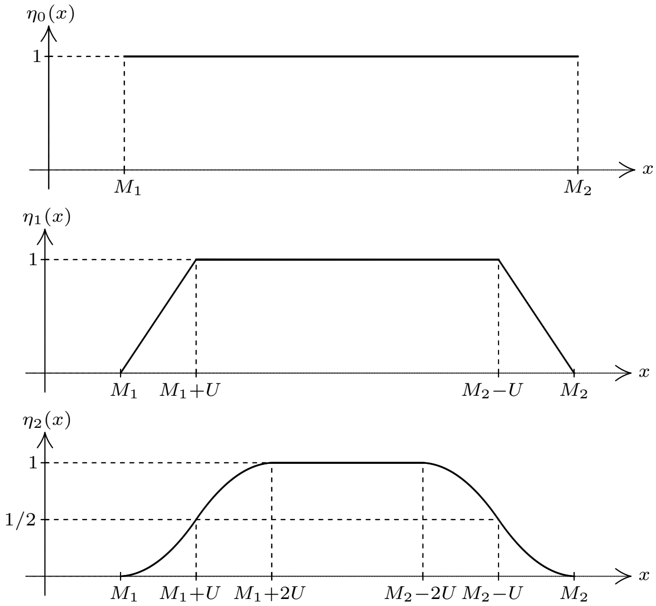

Figure 3. A sketch of the weight functions η_0, η_1 and η_2. Please note that for $J \geqslant 1$ the weight function η_J is C^∞-smooth except for the points $M_1 + \ell U$ and $M_2 - \ell U$, where $\ell \in \{0, 1, \ldots, J\}$, where it belongs only to C^{J-1}.

where the error is

$$\ll (M_2 - M_1)\left(1 + \mu^J U^{-J}\right) G\, e^{-A|\alpha|\mu - A\,F} + \left(1 + \chi(x_0)\, F^{1/2}\right) G\, \mu\, F^{-3/2}$$
$$+ U^{-J} \sum_{j=0}^{J} \left(E_J(M_1 + jU) + E_J(M_2 - jU)\right).$$

Here A is some positive real constant independent of f, g, α, and $[M_1, M_2]$, χ denotes the characteristic function of the set $]M_1, M_1 + JU[\, \cup\,]M_2 - JU, M_2[$, the symbol $E_J(x)$ stands for

$$E_J(x) = \frac{G}{\left(|f'(x) + \alpha| + f''(x)^{1/2}\right)^{J+1}},$$

and the factor $\xi_J(x_0)$ is as follows:

1. *If $M_1 + JU < x_0 < M_2 - JU$, then $\xi_J(x_0) = 1$.*
2. *If $M_1 < x_0 \leqslant M_1 + JU$, then*

$$\xi_J(x_0) = (J!\, U^J)^{-1} \sum_{j=0}^{j_1} \binom{J}{j} (-1)^j \cdot \sum_{0 \leqslant \nu \leqslant J/2} c_\nu\, f''(x_0)^{-\nu}\, (x_0 - M_1 - jU)^{J-2\nu},$$

   where j_1 is the largest integer with $M_1 + j_1 U < x_0$.

3. If $M_2 - JU \leqslant x_0 < M_2$, then

$$\xi_J(x_0) = (J!\,U^J)^{-1} \sum_{j=0}^{j_2} \binom{J}{j} (-1)^j \cdot \sum_{0 \leqslant \nu \leqslant J/2} c_\nu \, f''(x_0)^{-\nu} \left(M_2 - jU - x_0\right)^{J-2\nu},$$

where j_2 is the largest integer with $M_2 - j_2 U > x_0$.

The coefficients c_ν are fixed numerical constants only depending on J. Furthermore, $\xi_J(x)$ is continuously differentiable in the intervals $]M_1, M_1 + JU[$ and $]M_2 - JU, M_2[$ except for the points $M_1 + jU$ and $M_2 - jU$ appearing in the above sums, and the derivative satisfies in these intervals, where it exists, the estimate $\xi_J'(x) \ll \mu^{-1} + U^{-1}$.

Strictly speaking, the last statement about $\xi_J'(x)$ does not appear in the statement of Theorem 2.2 in [Ju4], but it follows easily by inspecting the above sums for $\xi_J(x_0)$.

Some of the exponential integrals we will meet will not have saddle points. They can be handled with the following theorem, which is a special case of Theorem 2.3 in [Ju4].

Theorem 11. *Let us consider an interval $[M_1, M_2] \subseteq \mathbb{R}_+$, let $\mu \in \mathbb{R}_+$, and let D stand for the domain*

$$D = \big\{ z \in \mathbb{C} : |z - x| < \mu \text{ for some } x \in [M_1, M_2] \big\}.$$

Let $f, g \colon D \longrightarrow \mathbb{C}$ be holomorphic, let $F, G \in \mathbb{R}_+$, and assume that $f(x) \in \mathbb{R}$ and $f'(x) \asymp F\,\mu^{-1}$ for $x \in [M_1, M_2]$, and that $f'(z) \ll F\,\mu^{-1}$ and $g(z) \ll G$ for $z \in D$.

Next, let $U \in \mathbb{R}_+$ and $J \in \mathbb{Z}_+$ be such that $2\,J\,U < M_2 - M_1$, and let η_J denote the weight function defined as above, namely the convolution

$$\eta_J = \frac{1}{U}\chi_{[0,U]} * \frac{1}{U}\chi_{[0,U]} * \cdots * \frac{1}{U}\chi_{[0,U]} * \chi_{[M_1, M_2 - JU]},$$

with $U^{-1}\chi_{[0,U]}$ appearing J times. Finally, let $\alpha \in \mathbb{R}$. Then

$$\int_{M_1}^{M_2} g(x)\, e(f(x) + \alpha x)\, \eta_J(x)\, \mathrm{d}x \ll U^{-J}\, G\, \mu^{J+1}\, F^{-J-1} + \left(\mu^J\, U^{1-J} + M_2 - M_1\right) G\, e^{-A\,F}.$$

Here A is some positive real constant independent of f, g, α, and $[M_1, M_2]$.

We will also use the following lemma for estimating exponential integrals. It is Lemma 6 in [JM].

Lemma 12. *Let $M_1, M_2 \in \mathbb{R}_+$ and $M_1 < M_2$, let $J \in \mathbb{Z}_+$, and let $g \in C_{\mathrm{c}}^J(\mathbb{R}_+)$ with $\operatorname{supp} g \subseteq [M_1, M_2]$, and let G_0 and G_1 be such that $g^{(\nu)}(x) \ll_\nu G_0\, G_1^{-\nu}$ for all $x \in \mathbb{R}_+$ for each $\nu \in \{0, 1, 2, \ldots, J\}$. Also, let f be a holomorphic function defined in $D \subseteq \mathbb{C}$, which consists all points in the complex plane with distance smaller than $\mu \in \mathbb{R}_+$ from the interval $[M_1, M_2]$ of the real axis. Assume that f is real-valued on $[M_1, M_2]$ and let $F_1 \in \mathbb{R}_+$ be such that $F_1 \ll |f'(z)|$ for all $z \in D$. Then, for $P \in \{1, 2, \ldots, J\}$,*

$$\int_{M_1}^{M_2} g(x)\, e(f(x))\, \mathrm{d}x \ll_P G_0\, (G_1\, F_1)^{-P} \left(1 + \frac{G_1}{\mu}\right)^P (M_2 - M_1)\,.$$

Finally, for completeness, we state the following two classical tools, known as the first derivative test and the second derivative test, respectively. For a discussion of these, see e.g. Section 5.1 in [H1].

Lemma 13. *Let $M_1, M_2 \in \mathbb{R}$ with $M_1 < M_2$, let $\lambda \in \mathbb{R}_+$, and let f be a real-valued continuously differentiable function on $]M_1, M_2[$ such that $|f'(x)| \geqslant \lambda$ for $x \in]M_1, M_2[$. Also, let g be a complex-valued continuously differentiable function on the interval $[M_1, M_2]$, and let $G \in \mathbb{R}_+$ be such that $g(x) \ll G$ for $x \in [M_1, M_2]$. Then*

$$\int_{M_1}^{M_2} g(x)\, e(f(x))\, \mathrm{d}x \ll \frac{G}{\lambda} + \frac{1}{\lambda} \int_{M_1}^{M_2} |g'(x)|\, \mathrm{d}x.$$

Lemma 14. *Let $M_1, M_2 \in \mathbb{R}$ with $M_1 < M_2$, let $\lambda \in \mathbb{R}_+$, and let f be a real-valued twice continuously differentiable function on $]M_1, M_2[$ such that $|f''(x)| \geqslant \lambda$ for $x \in]M_1, M_2[$. Also, let g be a complex-valued continuously differentiable function on the interval $[M_1, M_2]$, and let $G \in \mathbb{R}_+$ be such that $g(x) \ll G$ for $x \in [M_1, M_2]$. Then*

$$\int_{M_1}^{M_2} g(x)\, e(f(x))\, \mathrm{d}x \ll \frac{G}{\sqrt{\lambda}} + \frac{1}{\sqrt{\lambda}} \int_{M_1}^{M_2} |g'(x)|\, \mathrm{d}x.$$

4 A transformation formula for smoothed exponential sums

In the following theorem $\delta_1, \delta_2, \ldots$ denote positive constants which may be supposed to be arbitrarily small.

Theorem 15. *Let $2 \leqslant M_1 < M_2 \leqslant 2M_1$. We assume that M_1 is sufficiently large, the notion of sufficiently large depending on the implicit constants in the assumptions below and on δ_1. Let f and g be holomorphic functions in the domain*

$$D = \big\{ z \in \mathbb{C} : |z - x| < c\, M_1 \text{ for some } x \in [M_1, M_2] \big\},$$

where c is a positive constant. Suppose that $f(x)$ is real for $x \in [M_1, M_2]$. Suppose also that, for some positive numbers F and G, $g(z) \ll G$, and $f'(z) \ll F\, M_1^{-1}$ for $z \in D$, and that $f''(x) > 0$ and $f''(x) \gg F\, M_1^{-2}$ for $x \in [M_1, M_2]$.

Let $r = h/k$ be a rational number such that $(h, k) = 1$, $1 \leqslant k \ll M_1^{1/2-\delta_1}$, $r \asymp F\, M_1^{-1}$ and $f'(M(r)) = r$ for a certain number $M(r) \in]M_1, M_2[$. Write $M_j = M(r) + (-1)^j\, m_j$ for $j \in \{1, 2\}$. Suppose that $m_1 \asymp m_2$, and that

$$M_1^{1+\delta_2}\, F^{-1/2} \ll m_1 \ll M_1^{1-\delta_3}.$$

Define for $j \in \{1, 2\}$

$$p_{j,n}(x) = f(x) - rx + (-1)^{j-1} \left(\frac{2\sqrt{nx}}{k} - \frac{1}{8} \right), \quad \text{and} \quad n_j = (r - f'(M_j))^2\, k^2\, M_j,$$

and for $n < n_j$ let $x_{j,n}$ be the (unique) zero of $p'_{j,n}(x)$ in the interval $]M_1, M_2[$. Also, let J be a fixed positive integer and sufficiently large depending on δ_2 and δ_4. Let

$$U \gg F^{-1/2}\, M_1^{1+\delta_4} \asymp F^{1/2}\, r^{-1}\, M_1^{\delta_4},$$

where $\delta_4 > \delta_2$, and assume also that $JU < (M_2 - M_1)/2$. Write for $j \in \{1, 2\}$

$$M'_j = M_j + (-1)^{j-1}\, J\, U = M(r) + (-1)^j\, m'_j,$$

and suppose that $m_j' \asymp m_j$. Define

$$n_j' = (r - f'(M_j'))^2\, k^2\, M_j'.$$

Then we have

$$\sum_{M_1 \leqslant m \leqslant M_2} \eta_J(m)\, t(m)\, g(m)\, e(f(m)) \tag{6}$$

$$= i\, 2^{-1/2}\, k^{-1/2} \sum_{j=1}^{2} (-1)^{j-1} \sum_{n<n_j} w_j(n)\, t(n)\, e\left(\frac{-n\overline{h}}{k}\right) n^{-1/4}\, x_{j,n}^{-1/4}$$

$$\cdot\, g(x_{j,n})\, p_{j,n}''(x_{j,n})^{-1/2}\, e\left(p_{j,n}(x_{j,n}) + \frac{1}{8}\right)$$

$$+ O\left(k^{-1/2}\, m_1^{1/2}\, F^{-1}\, G\, |h|^{3/2}\, U\, F^{\vartheta}\, M_1^{\vartheta+\varepsilon}\right),$$

where

$$w_j(n) = 1 \quad \textit{for} \quad n < n_j', \quad \textit{and} \quad w_j(n) \ll 1 \quad \textit{for} \quad n < n_j,$$

$w_j(y)$ and $w_j'(y)$ are piecewise continuous functions in the interval $\left]n_j', n_j\right[$ with at most $J-1$ discontinuities, and

$$w_j'(y) \ll \left(n_j - n_j'\right)^{-1} \quad \textit{for} \quad n_j' < y < n_j$$

whenever $w_j'(y)$ exists.

Proof. This result is an analogue of Theorem 3.4 of [Ju4], the main difference being the extra factor $F^{\vartheta}\, M_1^{\vartheta+\varepsilon}$ in the error term. For space reasons, we do not give a proof of this result here, and merely indicate the most important differences to the proof of Theorem 3.4 of [Ju4]. A rather relevant observation is that the condition $M^{\delta_2}\, |hk| \ll m_1$, which appears in Theorem 3.4 of [Ju4], is actually not used in the proof. Similarly, we do not need the condition in the present Maass form setting either.

It turns out there is exactly one place in the proof which is sensitive to a non-zero value of ϑ, namely estimating the errors accrued from the second error term from Theorem 10. By examining the details, we see that the resulting sum over the range $n_j' \leqslant n < n_j$ suffers the extra factor $n_j^{\vartheta+\varepsilon}$, and it is straightforward to estimate $n_j^{\vartheta} \ll F^{\vartheta}\, M^{\vartheta}$.

One last remark is in order: if we assume that $t(n) \ll d(n)\, n^{\vartheta}$, then the factor M_1^{ε} in the error term may quite easily be replaced by $\log M_1$.

5 Estimates for non-linear sums

The savings in the estimates for short sums depend on an estimate for the kind of nonlinear sums that appear after the application of the Voronoi type summation formula. In the following theorem, it is essential that the estimate is better when shorter sums are considered.

Theorem 16. *Let $M \in [1, \infty[$, $\eta \in \mathbb{R}$, $B \in \mathbb{R}$, and $\Delta \in [1, M]$. Write $F = |B|\, M^{1/2}$, and assume that $M^2 \ll \Delta\, F$. Let g be a C^1-function on the interval $[M, M+\Delta]$ satisfying bounds $g(x) \ll G$ and $g'(x) \ll G'$ on $[M, M+\Delta]$ for some positive real numbers G and G'. Then*

$$\sum_{M \leqslant m \leqslant M+\Delta} t(m)\, g(m)\, e\left(\eta m + Bm^{1/2}\right) \ll \Delta^{5/6}\, (G + \Delta\, G')\, M^{\vartheta - 1/3}\, F^{1/3+\varepsilon}.$$

Proof. This is analogous to Theorem 4.1 in [EK], and in fact, the proof given in [EK] works almost verbatim in our case, except that now we use Theorem 15 instead of the corresponding result for holomorphic cusp form, i.e. Theorem 3.4 in [Ju4], and naturally, when smoothing error is to be estimated, an extra M^{ϑ} appears in a few places. There is only one point which requires extra clarification, the error term in Theorem 15 has the extra factor $F^{\vartheta} M^{\vartheta}$; this time the total error from the error terms coming from using the transformation formula contributes

$$\ll \frac{\Delta}{M} M^{1/2+\vartheta} F^{\vartheta+\varepsilon} \ll \Delta^{5/6} M^{\vartheta-1/3} F^{\vartheta+\varepsilon},$$

which is smaller than the desired upper bound.

It turns out that for long sums, the ϑ in the upper bound may be erased. This was proved by Karppinen in [Ka] by considering the mean value of the relevant exponential sums. Earlier, Jutila [Ju6, Ju7] had considered similar mean values for holomorphic cusp forms and the divisor function. The following estimate is Theorem 8.2 in [Ka].

Theorem 17. *Let $M \in [1, \infty[$, $\eta \in \mathbb{R}$, $B \in \mathbb{R}$, and $\Delta \in [1, M]$. Write $F = |B|\, M^{1/2}$, and assume that $M \ll F$. Let g be a C^1-function on the interval $[M, M+\Delta]$ satisfying the bounds $g(x) \ll G$ and $g'(x) \ll G'$ on $[M, M+\Delta]$ for some positive real numbers G and G'. Then*

$$\sum_{M \leqslant m \leqslant M+\Delta} t(m)\, g(m)\, e\Big(\eta m + B m^{1/2}\Big) \ll (G + \Delta\, G')\, M^{1/2}\, F^{1/3+\varepsilon}.$$

In fact, the proofs of our main theorems do not require Theorem 17 as we could use Theorem 16 instead. However, one of the upper bounds in Theorem 18 is better if Theorem 17 is applied instead of Theorem 16.

6 Proof of Theorem 1

We shall prove Theorem 1 by first proving estimates for smooth short exponential sums. For this purpose, we shall use a wide weight function $w \in C_{\mathrm{c}}^{\infty}(\mathbb{R}_+)$ taking only values from $[0, 1]$, supported in $[M, M+\Delta]$, and for which

$$w^{(\nu)}(x) \ll_{\nu} \Delta^{-\nu},$$

for every nonnegative integer ν. The following estimates give an analogue of Theorem 5.1 of [EK].

Theorem 18. *Let us be given a small $\varepsilon \in \mathbb{R}_+$, and let $\delta \in \mathbb{R}_+$ satisfy $\delta \ll \varepsilon$ with a sufficiently small implicit constant. Let $M \in [1, \infty[$, and let $\Delta \in [1, M]$ with $\Delta \gg M^{\delta}$. Furthermore, let $\alpha \in \mathbb{R}$, and let $h \in \mathbb{Z}$, $k \in \mathbb{Z}_+$ and $\eta \in \mathbb{R}$ be such that*

$$\alpha = \frac{h}{k} + \eta, \quad (h, k) = 1, \quad k \leqslant K, \quad |\eta| \leqslant \frac{1}{kK},$$

where $K = \Delta^{1/2-\delta}$.

1. *If $\eta \ll \Delta^{-1+\delta}$, then*

$$\sum_{M \leqslant n \leqslant M+\Delta} t(n)\, e(n\alpha)\, w(n) \ll_{\delta} \Delta^{1/6}\, M^{1/3+\varepsilon}.$$

2. *If $\Delta^{-1+\delta} \ll \eta$ and $k^2\, \eta^2\, M < 1/2$, then*

$$\sum_{M \leqslant n \leqslant M+\Delta} t(n)\, e(n\alpha)\, w(n) \ll_{\delta} 1.$$

3. If $\Delta^{-1+\delta} \ll \eta \ll M\,\Delta^{-2}$*,* $k^2\,\eta^2\,M \gg 1$ *and* $k^2\,\eta\,M\,\Delta^{-1+2\delta} \ll 1$*, then*

$$\sum_{M\leqslant n\leqslant M+\Delta} t(n)\,e(n\alpha)\,w(n) \ll_\delta 1 + k^{-1/2}\,\Delta\,M^{-1/4}\left(k^2\,\eta^2\,M\right)^{\vartheta-1/4+\varepsilon}.$$

4. If $\Delta^{-1+\delta} \ll \eta \ll M\,\Delta^{-2}$*,* $k^2\,\eta^2\,M \gg 1$ *and* $k^2\,\eta\,M\,\Delta^{-1+2\delta} \gg 1$*, then*

$$\sum_{M\leqslant n\leqslant M+\Delta} t(n)\,e(n\alpha)\,w(n) \ll_\delta \left(k^2\,\eta^2\,M\right)^{\vartheta}\Delta^{1/6}\,M^{1/3+\varepsilon}.$$

Remark. The proof of the estimate 1 employs Theorem 17. We could use Theorem 16 instead to obtain the upper bound $\ll \Delta^{1/6-\vartheta}\,M^{1/3+\vartheta+\varepsilon}$, which would be good enough to obtain the main theorems.

Proof. In this proof we consider ε to be fixed. We begin by applying the Voronoi summation formula to the sum under study. The proof will soon split into two cases depending on whether η is smaller than larger than $\Delta^{-1+\delta}$. Voronoi summation yields

$$\begin{aligned}&\sum_{M\leqslant n\leqslant M+\Delta} t(n)\,e(n\alpha)\,w(n)\\ &= \frac{\pi i}{k\,\sinh\pi\kappa}\sum_{n=1}^{\infty} t(n)\,e\!\left(\frac{-n\overline{h}}{k}\right)\int_M^{M+\Delta}(J_{2i\kappa}-J_{-2i\kappa})\!\left(\frac{4\pi\sqrt{nx}}{k}\right)e(\eta x)\,w(x)\,\mathrm{d}x\\ &\quad+\frac{4\cosh\pi\kappa}{k}\sum_{n=1}^{\infty} t(-n)\,e\!\left(\frac{n\overline{h}}{k}\right)\int_M^{M+\Delta}K_{2i\kappa}\!\left(\frac{4\pi\sqrt{nx}}{k}\right)e(\eta x)\,w(x)\,\mathrm{d}x.\end{aligned}$$

Using the asymptotics for the K-Bessel function, and picking some large $A\in\mathbb{R}_+$, the K-series can be estimated by

$$\ll_A \frac{1}{k}\sum_{n=1}^{\infty}|t(n)|\int_M^{M+\Delta}k^A\,n^{-A/2}\,x^{-A/2}\,w(x)\,\mathrm{d}x.$$

This is $\ll_A k^{A-1}\,\Delta\,M^{-A/2}$, provided that $A>2$, and since $k \ll M^{1/2-\delta}$, it is furthermore $\ll_\delta 1$, provided that $A \gg_\delta 1$. Similarly, by replacing the J-Bessel expression by the asymptotics given in (4) with some $K\in\mathbb{Z}_+$, the resulting O-terms contribute

$$\ll_K \frac{1}{k}\sum_{n=1}^{\infty}|t(n)|\int_M^{M+\Delta}k^{1/2+(K+1)}\,n^{-1/4-(K+1)/2}\,x^{-1/4-(K+1)/2}\,w(x)\,\mathrm{d}x,$$

and this is again $\ll_\delta 1$ for a fixed $K \gg_\delta 1$. Thus, we are led to

$$\begin{aligned}\sum_{M\leqslant n\leqslant M+\Delta} t(n)\,e(n\alpha)\,w(n) &= O(1)+\frac{C}{k}\sum_{n=1}^{\infty}t(n)\,e\!\left(\frac{-n\overline{h}}{k}\right)\\ &\cdot\int_M^{M+\Delta}\frac{k^{1/2}}{n^{1/4}\,x^{1/4}}\sum_{\pm}(\pm 1)\,e\!\left(\pm\frac{2\sqrt{nx}}{k}\right)e\!\left(\mp\frac{1}{8}\right)g_\pm(x;n,k)\,e(\eta x)\,w(x)\,\mathrm{d}x,\end{aligned}$$

where

$$g_\pm(x;n,k) = 1+\sum_{\ell=1}^{K}c_\ell^\pm\,k^\ell\,n^{-\ell/2}\,x^{-\ell/2},$$

and C is some real constant.

6.1 The case $\eta \ll \Delta^{-1+\delta}$ in the proof of Theorem 18

Write $X = k^2\, M\, \Delta^{3\delta-2}$. We shall handle separately the terms with $n > X$ and the terms with $n \leqslant X$. We shall also take the summation $\sum_{\pm}$ to the outside, and consider each choice of sign separately.

The high-frequency terms with $n > X$ contribute

$$\ll \frac{1}{k} \sum_{n>X} t(n)\, k^{1/2}\, n^{-1/4}\, e\biggl(\frac{-n\overline{h}}{k}\biggr) \cdot \int\limits_{M}^{M+\Delta} x^{-1/4}\, g_{\pm}(x;n,k)\, e\biggl(\pm\frac{2\sqrt{nx}}{k}+\eta x\biggr)\, w(x)\, \mathrm{d}x.$$

Since we now have $X^{1/2}\, k^{-1}\, M^{-1/2} \gg \Delta^{\delta/2}\, \eta$, Lemma 12 says that the integral here is

$$\int\limits_{M}^{M+\Delta} \ldots \mathrm{d}x \ll_P M^{-1/4} \left(\Delta\, n^{1/2}\, k^{-1}\, M^{-1/2}\right)^{-P} \Delta,$$

where $P \in \mathbb{Z}_+$. Provided that $P \geqslant 2$, the contribution from these high-frequency terms is

$$\begin{aligned}
&\ll_P \frac{1}{k} \sum_{n>X} |t(n)|\, \frac{k^{1/2}}{n^{1/4}}\, M^{-1/4} \left(\Delta\, n^{1/2}\, k^{-1}\, M^{-1/2}\right)^{-P} \Delta \\
&\ll k^{P-1/2}\, \Delta^{1-P}\, M^{P/2-1/4} \sum_{n>X} |t(n)|\, n^{-1/4-P/2} \\
&\ll_P k^{P-1/2}\, \Delta^{1-P}\, M^{P/2-1/4}\, X^{3/4-P/2},
\end{aligned}$$

and for a fixed $P \gg_\delta 1$, this is $\ll_\delta 1$.

Let us consider next the low-frequency terms with $n \leqslant X$. These contribute

$$\begin{aligned}
&\ll \frac{1}{k} \sum_{n\leqslant X} t(n)\, k^{1/2}\, n^{-1/4}\, e\biggl(\frac{-n\overline{h}}{k}\biggr) \\
&\quad \cdot \int\limits_{M}^{M+\Delta} x^{-1/4}\, g_{\pm}(x;n,k)\, e\biggl(\pm\frac{2\sqrt{nx}}{k}+\eta x\biggr)\, w(x)\, \mathrm{d}x \\
&= k^{-1/2} \sum_{\substack{L\leqslant X/2 \\ \text{dyadic}}} \int\limits_{M}^{M+\Delta} x^{-1/4}\, w(x) \\
&\quad \cdot \sum_{L<n\leqslant 2L} t(n)\, n^{-1/4}\, g_{\pm}(x;n,k)\, e\biggl(\pm\frac{2\sqrt{nx}}{k} - \frac{n\overline{h}}{k} + \eta x\biggr)\, \mathrm{d}x.
\end{aligned}$$

By Theorem 17, the conditions of which are met under the present circumstances, the sum $\sum_{L<n\leqslant 2L}$ can be estimated by

$$\ll L^{-1/4}\, L^{1/2} \left(M^{1/2}\, k^{-1}\, L^{1/2}\right)^{1/3+\varepsilon/6} \ll L^{5/12}\, M^{1/6+\varepsilon/2}\, k^{-1/3}.$$

Thus, the low-frequency terms contribute

$$\begin{aligned}
&\ll k^{-1/2} \sum_{\substack{L\leqslant X/2 \\ \text{dyadic}}} \Delta\, M^{-1/4}\, L^{5/12}\, M^{1/6+\varepsilon/2}\, k^{-1/3} \\
&\ll k^{-5/6}\, \Delta\, M^{-1/12+\varepsilon/2}\, X^{5/12} \ll \Delta^{1/6}\, M^{1/3+\varepsilon/2+5\delta/4},
\end{aligned}$$

and we are finished with the case $\eta \ll \Delta^{-1+\delta}$, provided that $\delta \leqslant 2\varepsilon/5$.

6.2 The case $\eta \gg \Delta^{-1+\delta}$ in the proof of Theorem 18

This time we will choose $X = k^2\,\eta^2\,M$. The high-frequency terms with $n > 2X$ are again handled in the same way as in the case $\eta \ll \Delta^{-1+\delta}$. For an integer $P \geqslant 2$, and for each choice of the sign $\pm$, they are

$$\begin{aligned}
&\ll \frac{1}{k}\sum_{n>2X} t(n)\,k^{1/2}\,n^{-1/4}\,e\!\left(\frac{-n\overline{h}}{k}\right)\\
&\qquad\cdot\int_M^{M+\Delta} x^{-1/4}\,g_\pm(x;n,k)\,e\!\left(\pm\frac{2\sqrt{nx}}{k}+\eta x\right)w(x)\,\mathrm{d}x\\
&\ll_P \frac{1}{k}\sum_{n>2X} |t(n)|\;k^{1/2}\,n^{-1/4}\,M^{-1/4}\left(\Delta\,n^{1/2}\,k^{-1}\,M^{-1/2}\right)^{-P}\Delta\\
&\ll_P k^{-1/2+P}\,\Delta^{1-P}\,M^{P/2-1/4}\,X^{3/4-P/2}.
\end{aligned}$$

For $P \gg_\delta 1$, this contribution is again $\ll_\delta 1$.

If $X < 1/2$, then the above already proves case 2 of the theorem, so let us assume that $X \gg 1$. The remaining terms, the ones with $n \leqslant 2X$, are then partitioned into two sets: those with $|n - X| \geqslant W$ and those with $|n - X| < W$, where $W = k^2\,M\,\eta\,\Delta^{-1+2\delta}$.

So, let us consider the terms with $n \leqslant 2X$ and $|n - X| \geqslant W$. The crucial observations here are that

$$\frac{\sqrt{X}}{k\sqrt{x}} - \frac{\sqrt{n}}{k\sqrt{x}} \asymp \frac{1}{k\sqrt{M}}\int_n^X \frac{\mathrm{d}t}{\sqrt{t}} \gg \frac{|n-X|}{k\sqrt{M}\sqrt{X}} \gg \frac{W}{k\sqrt{M}\sqrt{X}} = \Delta^{-1+2\delta},$$

and that, thanks to the assumption $\eta \ll M\,\Delta^{-2}$,

$$|\eta| - \frac{\sqrt{X}}{k\sqrt{x}} = \frac{\sqrt{X}}{k\sqrt{M}} - \frac{\sqrt{X}}{k\sqrt{x}} \asymp \frac{\sqrt{X}}{k}\int_M^x \frac{\mathrm{d}t}{t^{3/2}} \ll \frac{\sqrt{X}\,\Delta}{k\,M^{3/2}} = \frac{\eta\,\Delta}{M} \ll \Delta^{-1}.$$

Using these appropriately (depending on the sign of η), we conclude that

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\pm\frac{2\sqrt{nx}}{k}+\eta x\right) = \pm\frac{\sqrt{n}}{k\sqrt{x}} + \eta \gg \Delta^{-1+2\delta},$$

and so, by Lemma 12, the terms under consideration contribute

$$\begin{aligned}
&\ll_P \frac{1}{k}\sum_{\substack{n\leqslant 2X,\\ |n-X|\geqslant W}} |t(n)|\;k^{1/2}\,n^{-1/4}\,M^{-1/4}\,\Delta^{-2\delta P}\,\Delta\\
&\ll_P k^{-1/2}\,X^{3/4}\,M^{-1/4}\,\Delta^{-2\delta P}\,\Delta,
\end{aligned}$$

and for a fixed $P \gg_\delta 1$ this is again $\ll_\delta 1$.

Next, if $W \ll 1$, then the remaining terms, the ones with $|n - X| < W$, contribute

$$\ll k^{-1/2}\,X^{\vartheta-1/4+\varepsilon}\,\Delta\,M^{-1/4} \ll k^{-1/2}\,(k^2\eta^2 M)^{\vartheta-1/4+\varepsilon}\,\Delta\,M^{-1/4},$$

and we have established case 3. Finally, only case 4 remains.

So, let us assume that $W \gg 1$ and $W \leqslant X/3$. The idea now is to exchange integration and summation, apply Theorem 16 to the integrand with the parameters

$$M = X, \quad \Delta = W, \quad \text{and} \quad B = \frac{\sqrt{x}}{k},$$

observing that the condition $\Delta\, F \gg M^2$ of Theorem 16 holds, since it reduces to

$$W \cdot \frac{\sqrt{X}\sqrt{M}}{k} \gg X^2,$$

which further reduces to $k^2\, \eta^2\, \Delta^{1-2\delta} \ll 1$, which in turn follows from $|\eta| \leqslant 1/(kK)$. The remaining terms are then seen to contribute

$$\begin{aligned}
&\ll \frac{1}{k} \sum_{X-W<n<X+W} t(n)\, k^{1/2}\, n^{-1/4}\, e\biggl(\frac{-n\overline{h}}{k}\biggr) \\
&\quad \cdot \int\limits_{M}^{M+\Delta} x^{-1/4}\, g_\pm(x;n,k)\, e\biggl(\pm\frac{2\sqrt{nx}}{k} + \eta x\biggr)\, w(x)\, \mathrm{d}x \\
&\ll k^{-1/2} \int\limits_{M}^{M+\Delta} x^{-1/4}\, e(\eta x)\, w(x) \\
&\quad \cdot \sum_{X-W<n<X+W} t(n)\, n^{-1/4}\, g_\pm(x;n,k)\, e\biggl(-\frac{n\overline{h}}{k} \pm \frac{2\sqrt{nx}}{k}\biggr)\, \mathrm{d}x \\
&\ll k^{-1/2} \int\limits_{M}^{M+\Delta} x^{-1/4}\, w(x) \left(\frac{W}{X}\right)^{5/6} X^{-1/4}\, X^{1/2+\vartheta} \left(\frac{\sqrt{X}\sqrt{M}}{k}\right)^{1/3+\varepsilon/2} \mathrm{d}x \\
&\ll k^{-1/2}\, \Delta\, M^{-1/4} \left(\frac{k^2\, \eta\, M\, \Delta^{-1+2\delta}}{k^2\, \eta^2\, M}\right)^{5/6} \bigl(k^2\, \eta^2\, M\bigr)^{1/4+\vartheta} \left(\frac{k\, \eta\, M}{k}\right)^{1/3+\varepsilon/2} \\
&\ll k^{-1/2}\, \Delta\, M^{-1/4} \left(\frac{\Delta^{-1+2\delta}}{\eta}\right)^{5/6} k^{1/2+2\vartheta}\, \eta^{1/2+2\vartheta}\, M^{1/4+\vartheta}\, \eta^{1/3}\, M^{1/3+\varepsilon/2} \\
&\ll \bigl(k^2\, \eta^2\, M\bigr)^{\vartheta}\, \Delta^{1/6}\, M^{1/3+\varepsilon},
\end{aligned}$$

provided that $\delta \leqslant 3\varepsilon/10$.

Finally, we still need to consider the case $W > X/3$. However, in this case, we will simply split the summation range from $\max\{X-W, 1\}$ to $\min\{2X, X+W\}$ dyadically, and estimate each subsum exactly as in the above case $1 \ll W \leqslant X/3$.

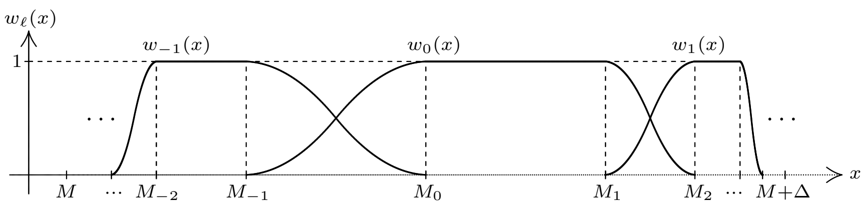

Figure 4. The weight functions $w_\ell(x)$ used in the proofs of Theorem 1, Proposition 19 and Lemma 31.

Proof of Theorem 1. We can now remove the weight function w from the estimates for short sums. For this purpose we shall introduce a partition of unity of

$]M, M+\Delta[$. Let us define a set of points M_ℓ for $\ell \in \mathbb{Z}$ by first setting

$$M_0 = M + \frac{\Delta}{2},$$

and then for each $\ell \in \mathbb{Z}_+$

$$M_{\pm\ell} = M + \frac{\Delta}{2} \pm \left(\frac{\Delta}{4} + \frac{\Delta}{8} + \ldots + \frac{\Delta}{2^{\ell+1}} \right).$$

We pick functions $w_\ell \in C_c^\infty(\mathbb{R})$ such that each w_ℓ only takes values from $[0,1]$, w_ℓ is supported on $[M_{2\ell-1}, M_{2\ell+2}]$, $w_\ell \equiv 1$ on $[M_{2\ell}, M_{2\ell+1}]$, and

$$w_\ell^{(\nu)}(x) \ll_\nu \left(\frac{\Delta}{4^{|\ell|}} \right)^{-\nu},$$

for $x \in [M_{2\ell-1}, M_{2\ell+2}]$ and $\nu \in \{0\} \cup \mathbb{Z}_+$, uniformly in ℓ. Furthermore, $w_\ell + w_{\ell+1}$ is to be $\equiv 1$ on $[M_{2\ell+1}, M_{2\ell+2}]$. Figure 4 depicts the situation.

Let now Δ be such that $\Delta^{3/2+\delta} \ll M$ for some $\delta \in \mathbb{R}_+$ with $\delta \ll \varepsilon$ with a sufficiently small implicit constant as in Theorem 18, and let $L \in \mathbb{Z}_+$ be such that

$$\Delta\, 4^{-L} \asymp M^{2/(5+6\vartheta)}.$$

Here we assume that $\Delta \gg M^{2/(5+6\vartheta)}$ for otherwise the desired estimate follows from a simple estimate by absolute values. Since $\Delta^{-1/2+\delta} \ll M\,\Delta^{-2}$, we have for any Farey approximation $\alpha = h/k + \eta$ of order $\Delta^{1/2-\delta}$ that

$$|\eta| \leqslant \frac{1}{k\,\Delta^{1/2-\delta}} \leqslant \frac{1}{\Delta^{1/2-\delta}} \ll \frac{M}{\Delta^2}.$$

Thus, we may apply Theorem 18 to get

$$\begin{aligned}\sum_{n\in\mathbb{Z}} t(n)\, e(n\alpha)\, w_\ell(n) &\ll_\delta \left(\frac{\Delta}{4^{|\ell|}} \right)^{1/6-\vartheta} M^{1/3+\vartheta+\varepsilon} + \frac{\Delta}{4^{|\ell|}} \cdot M^{-1/4} \\ &\ll \left(\frac{\Delta}{4^{|\ell|}} \right)^{1/6-\vartheta} M^{1/3+\vartheta+\varepsilon},\end{aligned}$$

uniformly in ℓ, and so

$$\begin{aligned}\sum_{\ell=-L}^{L} \sum_{n\in\mathbb{Z}} t(n)\, e(n\alpha)\, w_\ell(n) &\ll_\delta \sum_{\ell=-L}^{L} \left(\frac{\Delta}{4^{|\ell|}} \right)^{1/6-\vartheta} M^{1/3+\vartheta+\varepsilon} \\ &\ll \Delta^{1/6-\vartheta}\, M^{1/3+\vartheta+\varepsilon},\end{aligned}$$

and estimating by absolute values,

$$\sum_{M\leqslant n\leqslant M+\Delta} t(n)\, e(n\alpha) \left(1 - \sum_{\ell=-L}^{L} w_\ell(n) \right) \ll M^{2/(5+6\vartheta)}\, M^{\vartheta+\varepsilon} \ll \Delta^{1/6-\vartheta}\, M^{1/3+\vartheta+\varepsilon}.$$

Proposition 19. *Let us be given a small $\varepsilon \in \mathbb{R}_+$, and let $\delta \in \mathbb{R}_+$ satisfy $\delta \ll \varepsilon$ with a sufficiently small implicit constant. Let $M \in [1,\infty[$ and let $\Delta \in [1, M]$ satisfy $\Delta \gg M^{2/3}$. Also, let $\alpha \in \mathbb{R}$ have a rational approximation $\alpha = h/k + \eta$, where h and k are coprime integers with $1 \leqslant k \ll M^{1/3-\delta}$, and where $\eta \in \mathbb{R}$ satisfies $\eta \ll k^{-1}\,\Delta^{\delta-1/2}$ and $\eta \ll M\,\Delta^{-2}$. Then*

$$\sum_{M\leqslant n\leqslant M+\Delta} t(n)\, e(n\alpha) \ll_\delta \Delta^{1/6-\vartheta}\, M^{1/3+\vartheta+\varepsilon} + k^{-1/2}\, \Delta\, M^{-1/4} \left(k^2\, \eta^2\, M \right)^{\vartheta-1/4+\varepsilon},$$

where the second term on the right-hand side can be deleted unless $\Delta^{-1+\delta} \ll \eta$ and $k^2\,\eta\, M\,\Delta^{-1+2\delta} \ll 1 \ll k^2\,\eta^2\, M$.

Proof. This is very similar to the proof of Theorem 1 above. In particular, we may use the same weight functions w_ℓ, and we simply have an extra term on the right-hand side in the case indicated above.

We also need a slightly more complicated version:

Proposition 20. *Let us be given a small $\varepsilon \in \mathbb{R}_+$, and let $\delta \in \mathbb{R}_+$ satisfy $\delta \ll \varepsilon$ with a sufficiently small implicit constant. Let $M \in [1, \infty[$, and let $\Delta \in [1, M]$ with $\Delta \gg M^{2/3}$. Also, let $\alpha \in \mathbb{R}$ have rational approximations*

$$\alpha = \frac{h_0}{k_0} + \eta_0 = \frac{h_1}{k_1} + \eta_1 = \ldots = \frac{h_L}{k_L} + \eta_L,$$

where $L \in \mathbb{Z}_+$ is chosen so that $\Delta\, 4^{-L} \asymp M^{2/3}$, and that $h_0, h_1, \ldots, h_L \in \mathbb{Z}$, $k_0, k_1, \ldots, k_L \in \mathbb{Z}_+$ with

$$k_0 \leqslant \Delta^{1/2-\delta}, \quad k_1 \leqslant \left(\frac{2\Delta}{4}\right)^{1/2-\delta}, \quad \ldots, \quad k_L \leqslant \left(\frac{2\Delta}{4^L}\right)^{1/2-\delta},$$

and $\eta_0, \eta_1, \ldots, \eta_L \in \mathbb{R}$ with

$$|\eta_0| \leqslant k_0^{-1}\, \Delta^{\delta-1/2}, \quad |\eta_1| \leqslant k_1^{-1} \left(\frac{2\Delta}{4}\right)^{\delta-1/2}, \quad \ldots, \quad |\eta_L| \leqslant k_L^{-1} \left(\frac{2\Delta}{4^L}\right)^{\delta-1/2},$$

and assume furthermore that $\eta_\ell \ll M\, (\Delta\, 4^{-\ell})^{-2}$ for each $\ell \in \{0, 1, 2, \ldots, L\}$ for which $\left(\Delta\, 4^{-\ell}\right)^{-1+\delta} \ll \eta$ and $k_\ell^2\, \eta_\ell^2\, M \gg 1$. Then

$$\sum_{M \leqslant n \leqslant M+\Delta} t(n)\, e(n\alpha) \ll_\delta \Delta^{1/6-\vartheta}\, M^{1/3+\vartheta+\varepsilon} + \sum_{\ell=1}^{L} k_\ell^{-1/2} \cdot \frac{\Delta}{4^\ell} \cdot M^{-1/4} \left(k_\ell^2\, \eta_\ell^2\, M\right)^{\vartheta-1/4+\varepsilon}.$$

Furthermore, here the term corresponding to a given value of ℓ can be deleted unless $\left(\Delta/4^\ell\right)^{-1+\delta} \ll \eta$ and $k_\ell^2\, \eta_\ell\, M \left(\Delta/4^\ell\right)^{-1+2\delta} \ll 1 \ll k_\ell^2\, \eta_\ell^2\, M$.

Proof. Again, the proof is very much similar to the proof of Theorem 1, but in this case each subsum $\sum t(n)\, e(n\alpha)\, w_\ell(n)$ is estimated with a Farey approximation $\alpha = h_{|\ell|}/k_{|\ell|} + \eta_{|\ell|}$ appropriate for the length of the support of w_ℓ, which is $\gg \Delta\, 4^{-|\ell|}$, and it is $\leqslant \Delta\, 4^{-|\ell|}$ for $\ell \geqslant 0$ and $\leqslant 2\, \Delta\, 4^{-|\ell|}$ for $\ell < 0$.

Proof of Theorem 3. These estimates follow from the proofs of Theorems 5.5 and 5.7 in [EK], except that Theorem 5.1 should be modified a little [E7]. The term $k^{-1}\, \Delta\, |\eta|^{-1/2}\, M^{-1/2+\varepsilon}$ only appears in the case in which $k^2\, \eta^2\, M \gg 1$, in which case the relevant estimate (on p. 27 of [EK]) is actually

$$\begin{aligned}\widetilde{A}(M, \Delta, \alpha) &\ll \Delta^{1/6}\, M^{1/3+\varepsilon} + k^{-1/2}\, \Delta\, M^{-1/4} \left(k^2\, \eta^2\, M\right)^{\varepsilon-1/4} \\ &\ll \Delta^{1/6}\, M^{1/3+\varepsilon} + k^{-1/2}\, \Delta\, M^{-1/4}.\end{aligned}$$

Thus, the second upper bound of Theorem 5.1 is actually

$$\sum_{M \leqslant n \leqslant M+\Delta} a(n)\, e(n\alpha)\, w(n) \ll \Delta^{1/6}\, M^{1/3+\varepsilon} + k^{-1/2}\, \Delta\, M^{-1/4+\varepsilon}.$$

7 Proof of Theorem 4 and Corollary 5

Let $U \in [1, \Delta/2[$. We shall pick a weight function $w \in C_c^2(\mathbb{R}_+)$ taking only non-negative real values, supported in $[M, M+\Delta]$, identically equal to 1 in the interval $[M+U, M+\Delta-U]$, for which

$$w^{(\nu)}(x) \ll U^{-\nu},$$

for each $\nu \in \{0, 1, 2\}$, and whose derivatives are supported in the set $[M, M+U] \cup [M+\Delta-U, M+\Delta]$. Sums with this weight function can be estimated rather nicely:

Lemma 21. *Let $X \in \mathbb{R}_+$ with $X \gg 1$, let $M \in [1, \infty[$, and let U and w be as above. Also, let h and k be coprime integers with $1 \leqslant k \ll M^{1/2-\delta}$, where δ is a fixed positive real number. Then*

$$\sum_{M \leqslant n \leqslant M+\Delta} t(n)\, e\left(\frac{nh}{k}\right) w(n) \ll_\delta k^{1/2}\, X^{1/4}\, M^{1/4} + k^{3/2}\, X^{-1/4}\, M^{3/4}\, U^{-1}.$$

Furthermore, if we select $X = 1/2$, we can forget the first term on the right-hand side of the estimate.

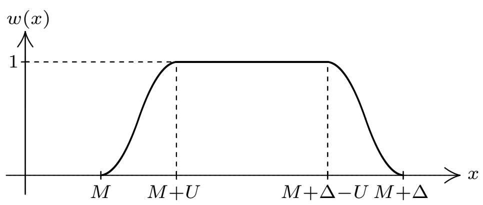

Figure 5. The weight function $w(x)$ of Lemma 21.

We first show how Lemma 21 is used to prove Theorem 4, and only then present the proof of Lemma 21.

Proof of Theorem 4. Introducing the above weight function w gives

$$\sum_{M \leqslant n \leqslant M+\Delta} t(n)\, e\left(\frac{nh}{k}\right) \ll U\, M^{\vartheta+\varepsilon} + \sum_{M \leqslant n \leqslant M+\Delta} t(n)\, e\left(\frac{nh}{k}\right) w(n).$$

If we select $U = k^{2/3}\, M^{1/3-2\vartheta/3}$ and $X = k^{2/3}\, M^{1/3+4\vartheta/3}$ in Lemma 21, we obtain

$$\begin{aligned}\sum_{M \leqslant n \leqslant M+\Delta} t(n)\, e\left(\frac{nh}{k}\right) & \\ &\ll U\, M^{\vartheta+\varepsilon} + k^{1/2}\, X^{1/4}\, M^{1/4} + k^{3/2}\, X^{-1/4}\, M^{3/4}\, U^{-1} \\ &\ll k^{2/3}\, M^{1/3+\vartheta/3+\varepsilon},\end{aligned}$$

as required.

When $M^{3/(5+6\vartheta)-1/2+\vartheta} \ll k \ll M^{5/18+\vartheta/3}$ we argue similarly, except that now the smoothing error is $\ll U^{1/6-\vartheta}\, M^{1/3+\vartheta+\varepsilon}$ by Theorem 1, and we choose $X = k^2\, M\, U^{-2}$ and $U = k^{3/(2-3\vartheta)}\, M^{(1-6\vartheta)/(4-6\vartheta)}$. We observe that Theorem 1 is applicable here since a little simplification shows that

$$U \ll \left(M^{5/18+\vartheta/3}\right)^{3/(2-3\vartheta)} \cdot M^{(1-6\vartheta)/(4-6\vartheta)} = M^{2/3}.$$

This choice of X and U leads to

$$\begin{aligned}
&\sum_{n\leqslant M} t(n)\, e\Big(\frac{nh}{k}\Big)\\
&\qquad \ll U^{1/6-\vartheta}\, M^{1/3+\vartheta+\varepsilon} + k^{1/2}\, X^{1/4}\, M^{1/4} + k^{3/2}\, X^{-1/4}\, M^{3/4}\, U^{-1}\\
&\qquad \ll k^{(1-6\vartheta)/(4-6\vartheta)}\, M^{3/(8-12\vartheta)+\varepsilon},
\end{aligned}$$

as required.

When $M^{5/18+\vartheta/3} \ll k \ll M^{1/2-\varepsilon}$ we choose the parameters to be $X = k^2\, M\, U^{-2}$ and $U = k^{2/3}\, M^{13/27-2\vartheta/9}$, so that $U \gg M^{2/3}$, and get

$$\begin{aligned}
&\sum_{n\leqslant M} t(n)\, e\Big(\frac{nh}{k}\Big)\\
&\qquad \ll U\, M^{\vartheta/3-2/9+\varepsilon} + k^{1/2}\, X^{1/4}\, M^{1/4} + k^{3/2}\, X^{-1}\, M^{3/4}\, U^{-1}\\
&\qquad \ll k^{2/3}\, M^{7/27+\vartheta/9+\varepsilon}.
\end{aligned}$$

Proof of Lemma 21. We shall feed the sum in question to the Voronoi type summation formula cited in Theorem 9 with the choice $f = w$. The series involving the K-Bessel function will be negligible: Pick any $A \in\,]2,\infty[$. Then the series involving the K-Bessel function can be estimated to be

$$\begin{aligned}
&\ll_A \frac{1}{k}\sum_{n=1}^{\infty} |t(n)| \int_M^{M+\Delta} w(x)\, k^A\, n^{-A/2}\, x^{-A/2}\, \mathrm{d}x\\
&\ll_A \frac{1}{k}\,\Delta\,(k\, M^{-1/2})^A \ll \frac{1}{k}\,\Delta\, M^{-\delta A}.
\end{aligned}$$

For $A \gg_{\delta,B} 1$, this is $\ll_B M^{-B}$ for arbitrary $B\in\mathbb{R}_+$.

In the series involving the J-Bessel function, we apply (4) with $K = 1$. The terms involving the error term contribute only

$$\begin{aligned}
&\ll \frac{1}{k}\sum_{n=1}^{\infty} |t(n)| \int_M^{M+\Delta} w(x)\, k^{5/2}\, n^{-5/4}\, x^{-5/4}\, \mathrm{d}x\\
&\ll k^{3/2}\,\Delta\, M^{-5/4} \ll k^{1/2}\, M^{1/4} \ll k^{1/2}\, X^{1/4}\, M^{1/4}.
\end{aligned}$$

In the case $X = 1/2$, we also have $k^{3/2}\,\Delta\, M^{-5/4} \ll k^{3/2}\, M^{3/4}\, U^{-1}$.

We shall consider the series involving the J-function in two parts according to whether $n \leqslant X$ or $n > X$. The high-frequency terms $n > X$ are again treated by integrating by parts twice. However, here there will be a slight twist: the bound for the integral

$$\int_M^{M+\Delta} w(x)\, k^{1/2}\, n^{-1/4}\, x^{-1/4}\, \Big(1 + C_\pm\, k\, n^{-1/2}\, x^{-1/2}\Big)\, e\Big(\pm\frac{2\sqrt{nx}}{k}\Big)\, \mathrm{d}x$$

will be

$$\ll k^{5/2}\, n^{-5/4}\, M^{3/4}\, \big(M^{-2}\,\Delta + U^{-1}\big) \ll k^{5/2}\, n^{-5/4}\, M^{3/4}\, U^{-1},$$

instead of $\ll k^{5/2}\, n^{-5/4}\, M^{3/4}\, U^{-2}\,\Delta$. The reason for this is that after having integrated by parts twice, the resulting integral is estimated by absolute values, and most of the terms in the integrands will be supported on $\operatorname{supp} w'$ which is a set of length

$\ll U$. The only terms in which the integrand is supported in a larger set are those, which still feature $w(x)$ after differentiation, but here the other factors all give an extra M^{-1} instead of mere U^{-1} upon differentation.

Substituting the bound from integration by parts back into the series, we see that the contribution from the high-frequency terms is

$$\ll \frac{1}{k}\sum_{n>X}|t(n)|\,k^{5/2}\,n^{-5/4}\,M^{3/4}\,U^{-1}\ll k^{3/2}\,X^{-1/4}\,M^{3/4}\,U^{-1}.$$

With the low-frequency terms, we estimate the integral in question by the first derivative test to get

$$\int\limits_M^{M+\Delta}\ldots\,\mathrm{d}x\ll k^{1/2}\,n^{-1/4}\,M^{-1/4}\,\frac{k\sqrt{M}}{\sqrt{n}},$$

and so the contribution from the low-frequency terms is

$$\ll \frac{1}{k}\sum_{n\leqslant X}|t(n)|\,k^{1/2}\,n^{-1/4}\,M^{-1/4}\,k\,M^{1/2}\,n^{-1/2}\ll k^{1/2}\,X^{1/4}\,M^{1/4}.$$

Proof of Corollary 5. This is proved in exactly the same way as the corresponding result, Theorem 1, in [V], except that Theorem 3 above is used to estimate the smoothing error. The only change in the computations when $M^{1/4}\ll k\ll M^{5/18}$ is to observe that when $k\ll M^{5/18}$, we have $k^{3/2}\,M^{1/4}\ll M^{2/3}$. When $k\gg M^{5/18}$, we choose $U=k^{2/3}\,M^{13/27}$ instead of $U=k^{2/3}\,M^{11/24}$, and we observe that for these choices we have the required lower bound $U\gg M^{2/3}$.

8 Proof of Theorem 6

Let $J\in\mathbb{Z}_+$. In order to be able to apply the Voronoi summation formula, we shall consider the smoothed exponential sum

$$\sum_{M_{-1}\leqslant n\leqslant M_2} t(n)\,w(n)\,e(\alpha n), \tag{7}$$

where w is the weight function η_J (see Section 3) which corresponds to the interval $[M_{-1},M_2]$ with parameter $U\in\mathbb{R}_+$, which is defined as follows: Let $d\in\mathbb{R}_+$ be a small constant depending on ε, and write $U=M^{1/2}\,\eta^{-1/2}\left(k^2\,\eta^2\,M\right)^d$. Let $M_{-1}=M-JU$, $M_1=M+\Delta$, and $M_2=M+\Delta+JU$. Also, we define $N_i=k^2\,\eta^2\,M_i$ for $i\in\{-1,1,2\}$ and $N=k^2\,\eta^2\,M$. The choice of J depends on d, and a fortiori on ε.

We record here a few simple observations about the paratemer U. First, since $1\ll k^2\,\eta^2\,M$ and $\eta\ll k^{-1}\,M^{-1/4}\ll 1$, we certainly have $U\gg M^{1/2}$. We also have $\eta^{-1/2}\ll k^{1/2}\,M^{1/4}\ll M^{3/8}$, and $k^2\,\eta^2\,M\ll k^2\,k^{-2}\,M^{-1/2}\,M=M^{1/2}$, and so $U\ll M^{7/8+d/2}$, and this is $o(M)$ for sufficiently small d.

8.1 Estimating smoothing error

First, we estimate the error caused by the introduction of the weight function w.

Lemma 22. *Let $M\in[1,\infty[$, and let $\alpha\in\mathbb{R}$, $h\in\mathbb{Z}$, $k\in\mathbb{Z}_+$ and $\eta\in\mathbb{R}$ be such that*

$$\alpha=\frac{h}{k}+\eta,\quad k\leqslant M^{1/4},\quad (h,k)=1\quad\textit{and}\quad |\eta|\leqslant\frac{1}{k\,M^{1/4}}.$$

Furthermore, let $U\in\mathbb{R}_+$ with $U\asymp M^{1/2}\,\eta^{-1/2}\left(k^2\,\eta^2\,M\right)^d$, where $d\in\mathbb{R}_+$. Then, given $\varepsilon\in\mathbb{R}_+$, we have

$$\sum_{M\leqslant n\leqslant M+U} t(n)\,e(n\alpha)\ll M^{1/2}\left(k^2\,\eta^2\,M\right)^{\vartheta/2-1/12+\varepsilon},$$

for any fixed $d \ll_\varepsilon 1$.

Now, by partial summation and Lemma 22 we have

$$\sum_{M_{-1} \leqslant n < M} t(n)\, e(\alpha n)\, w(n) + \sum_{M_1 < n \leqslant M_2} t(n)\, e(\alpha n)\, w(n)$$
$$\ll M^{1/2} \left(k^2\, \eta^2\, M\right)^{\vartheta/2 - 1/12 + \varepsilon}. \tag{8}$$

Proof of Lemma 22. Let us first dispose of the case $U \ll M^{2/3}$. In this case we have, by Theorem 1,

$$\sum_{M \leqslant n \leqslant M+U} t(n)\, e(n\alpha) \ll U^{1/6-\vartheta}\, M^{1/3+\vartheta+\varepsilon/8}$$
$$\ll \left(M^{1/2}\, \eta^{-1/2} \left(k^2\, \eta^2\, M\right)^d\right)^{1/6-\vartheta} M^{1/3+\vartheta+\varepsilon/8}$$
$$\ll M^{1/2} \left(k^2\, \eta^2\, M\right)^{\vartheta/2-1/12+d/6-d\vartheta}\ M^{\varepsilon/8}\, k^{1/6-\vartheta}\, \eta^{1/12-\vartheta/2}.$$

If $k^2\, \eta^2\, M \gg M^{1/4}$, then certainly

$$M^{\varepsilon/8}\, k^{1/6-\vartheta}\, \eta^{1/12-\vartheta/2} \leqslant M^{\varepsilon/8} \ll \left(k^2\, \eta^2\, M\right)^{\varepsilon/2}.$$

If $k^2\, \eta^2\, M \ll M^{1/4}$, then

$$k^{1/12-\vartheta/2}\, \eta^{1/12-\vartheta/2} \ll \left(M^{-3/8}\right)^{1/12-\vartheta/2} = M^{-1/32+3\vartheta/16},$$

so that

$$M^{\varepsilon/8}\, k^{1/6-\vartheta}\, \eta^{1/12-\vartheta/2} \ll M^{\varepsilon/8}\, k^{1/12-\vartheta/2}\, M^{-1/32+3\vartheta/16}$$
$$\ll M^{\varepsilon/8+1/48-1/32-\vartheta/8+3\vartheta/16} \ll 1.$$

Thus, in either case the sums of length $U \ll M^{2/3}$ are sufficiently small. The same argument also takes care of all the later terms which have the shape $U^{1/6-\vartheta}\, M^{1/3+\vartheta+\varepsilon}$.

Let us next focus on the case $U \gg M^{2/3}$. Let us first assume that h/k is a Farey fraction of order $U^{1/2-\delta}$ for some small $\delta \in \mathbb{R}_+$, sufficiently small depending on ε, i.e. that $|\eta| \leqslant k^{-1}\, U^{-1/2+\delta}$. Here the requirement for sufficiently small also depends on the relevant conditions in Propositions 19 and 20. We also assume that d is small enough depending on δ.

We split our sum of length U to subsums of length $\asymp \widetilde{U}$, where we take $\widetilde{U} = c\, U \left(k^2\, \eta^2\, M\right)^{-d} = c\, M^{1/2}\, \eta^{-1/2}$, where $c \in \mathbb{R}_+$ is a small fixed constant so that we can be sure that $\widetilde{U} \leqslant U$. We get $\asymp \left(k^2\, \eta^2\, M\right)^d$ subsums of lengths satisfying both $\gg \widetilde{U}$ and $\leqslant \widetilde{U}$ this way. We wish to apply Proposition 19 to each of our subsums. This purpose in mind, let us first observe that $|\eta| \leqslant k^{-1}\, \widetilde{U}^{-1/2+\delta}$, and that $\eta \ll M\, \widetilde{U}^{-2}$, since this simplifies to $1 \ll c^{-2}$, which holds by choosing c sufficiently small. Thus, we may apply Proposition 19. The contribution from the first term of the upper bound from Proposition 19 is $\ll M^{1/2}\, \left(k^2\, \eta^2\, M\right)^{\vartheta/2-1/12}$ in the same way as above, and since there are $\asymp \left(k^2\, \eta^2\, M\right)^d$ such terms, the total contribution is $\ll M^{1/2} \left(k^2\, \eta^2\, M\right)^{\vartheta/2-1/12+d}$. For each subsum the second term from Proposition 19 contributes

$$\ll k^{-1/2}\, \widetilde{U}\, M^{-1/4} \left(k^2\, \eta^2\, M\right)^{\vartheta-1/4+\varepsilon/2},$$

and we again get $\asymp \left(k^2\,\eta^2\,M\right)^d$ such terms, leading to a total contribution

$$\begin{aligned}
&\ll k^{-1/2}\,U\,M^{-1/4}\left(k^2\,\eta^2\,M\right)^{\varepsilon/2+\vartheta-1/4}\\
&\ll k^{-1/2}\,M^{1/2}\,\eta^{-1/2}\left(k^2\,\eta^2\,M\right)^{d}M^{-1/4}\left(k^2\,\eta^2\,M\right)^{\varepsilon/2+\vartheta-1/4}\\
&\ll M^{1/2}\left(k^2\,\eta^2\,M\right)^{d+\varepsilon/2+\vartheta/2-1/12}\left(k^2\,\eta^2\,M\right)^{\vartheta/2-5/12}\\
&\ll M^{1/2}\left(k^2\,\eta^2\,M\right)^{d+\varepsilon/2+\vartheta/2-1/12}.
\end{aligned}$$

Let us observe next that if $U \gg M^{5/6}$, then $M^{5/4} \ll U^{3/2}$. Further,

$$k\,M\left(k^2\,\eta^2\,M\right)^{2d} \ll U^{3/2+\delta},$$

for sufficiently small d, so that

$$\eta \asymp M\,U^{-2}\left(k^2\,\eta^2\,M\right)^{2d} \ll \frac{1}{k\,U^{1/2-\delta}}.$$

Thus, if $U \gg M^{5/6}$, then h/k is indeed a Farey fraction of order $U^{1/2-\delta}$ and everything is fine.

The remaining length range is $M^{2/3} \ll U \ll M^{5/6}$, and the only problematic case is the one in which $\eta \gg k^{-1}\,U^{\delta-1/2}$. In this case we use Proposition 20 which involves many Farey approximations possibly different from h/k. Let us consider one such Farey approximation $\alpha = h_\ell/k_\ell + \eta_\ell$, where $\ell \in \{0,1,2,\ldots,L\}$, $L \in \mathbb{Z}_+$, $U\,4^{-L} \asymp M^{2/3}$, and $h_\ell \in \mathbb{Z}$, $k_\ell \in \mathbb{Z}_+$, $\eta_\ell \in \mathbb{R}$, $(h_\ell,k_\ell)=1$, $k_\ell \leqslant \left(\nu_\ell\,U\,4^{-\ell}\right)^{1/2-\delta/2}$, and $|\eta| \leqslant k_\ell^{-1}\,(\nu_\ell\,U\,4^{-\ell})^{\delta/2-1/2}$, where ν_ℓ is 1 or 2, depending on whether $\ell = 0$ or $\ell \neq 0$, respectively. We first treat in detail the case in which $h_\ell/k_\ell \neq h/k$ for each $\ell \in \{0,1,\ldots,L\}$.

Let us observe that if we had $k_\ell \leqslant M^{1/4}/2$, then we would have

$$\frac{1}{k\,k_\ell} \leqslant \left|\frac{h_\ell}{k_\ell}-\frac{h}{k}\right| \leqslant |\eta|+|\eta_\ell| \leqslant \frac{1}{k\,M^{1/4}}+\frac{1}{k_\ell\,(U\,4^{-\ell})^{1/2-\delta/2}},$$

so that

$$1 \leqslant k_\ell\,M^{-1/4}+k\left(\frac{U}{4^\ell}\right)^{\delta/2-1/2} \leqslant \frac{1}{2}+M^{1/4+\varepsilon-1/3} = \frac{1}{2}+M^{\varepsilon-1/12} = \frac{1}{2}+o(1),$$

which is impossible. Thus, we must have $k_\ell \gg M^{1/4}$.

Let us now consider the term $k_\ell^{-1/2}\,U\,4^{-\ell}\,M^{-1/4}\left(k_\ell^2\,\eta_\ell^2\,M\right)^{\vartheta-1/4+\varepsilon}$ which only arises when

$$\left(U\,4^{-\ell}\right)^{-1+\delta/2} \ll \eta_\ell \quad\text{and}\quad k_\ell^2\,\eta_\ell\,M\,(U\,4^{-\ell})^{-1+\delta} \ll 1 \ll k_\ell^2\,\eta_\ell^2\,M.$$

Let us check that the condition $\eta_\ell \ll M\,(U\,4^{-\ell})^{-2}$ required in this case holds. Namely, since

$$k_\ell^2\,\eta_\ell\,M\,(U\,4^{-\ell})^{-1+\delta} \ll 1,$$

we have

$$\eta_\ell \ll \frac{(U\,4^{-\ell})^{1-\delta}}{k_\ell^2\,M}.$$

It is therefore enough that

$$\frac{(U\,4^{-\ell})^{1-\delta}}{k_\ell^2\,M} \ll \frac{M}{(U\,4^{-\ell})^2},$$

i.e. that $(U\,4^{-\ell})^{3-\delta} \ll k_\ell^2\, M^2$. Since $k_\ell \gg M^{1/4}$, it is enough that $U\,4^{-\ell} \ll M^{5/6}$, which is indeed true.

Next, we need to check that the term $k_\ell^{-1/2}\, U\,4^{-\ell}\, M^{-1/4}\, \bigl(k_\ell^2\,\eta_\ell^2\, M\bigr)^{\vartheta-1/4+\varepsilon}$ is small enough. We have

$$\begin{aligned} k_\ell^{-1/2}\, U\,4^{-\ell}\, M^{-1/4}\, \bigl(k_\ell^2\,\eta_\ell^2\, M\bigr)^{\vartheta} &\ll M^{-1/8}\, U\,4^{-\ell}\, M^{-1/4}\, (U\,4^{-\ell})^{-\vartheta+\delta\vartheta}\, M^{\vartheta} \\ &\ll (U\,4^{-\ell})^{1-\vartheta}\, M^{\vartheta-3/8+\varepsilon}. \end{aligned}$$

We have

$$(U\,4^{-\ell})^{1-\vartheta}\, M^{\vartheta-3/8} \ll (U\,4^{-\ell})^{1/6-\vartheta}\, M^{1/3+\vartheta}$$

if and only if $(U\,4^{-\ell})^{5/6} \ll M^{17/24}$, or equivalently, $U\,4^{-\ell} \ll M^{17/20}$. But this holds, since $5/6 < 17/20$.

We still need to check that the condition $\eta_\ell \ll M\,(U\,4^{-\ell})^{-2}$ holds also in the case in which $k_\ell^2\,\eta_\ell^2\, M \gg 1$ and $k_\ell^2\,\eta_\ell\, M\,(U\,4^{-\ell})^{-1+\delta} \gg 1$. Since we have $U \asymp M^{1/2}\,\eta^{-1/2}\,\bigl(k^2\,\eta^2\,M\bigr)^{d}$, the question is, whether

$$\eta_\ell \ll \eta\,\bigl(k^2\,\eta^2\,M\bigr)^{-2d}\, 4^{2\ell}?$$

If this was not the case, then we could estimate

$$\begin{aligned} \eta_\ell &\gg \eta\,\bigl(k^2\,\eta^2\,M\bigr)^{-2d}\, 4^{2\ell} \gg k^{-1}\, U^{\delta-1/2}\,\bigl(k^2\,\eta^2\,M\bigr)^{-2d}\, 4^{2\ell} \\ &\gg k_\ell^{-1}\left(\frac{U}{4^\ell}\right)^{\delta/2-1/2} U^{\delta/2}\,\bigl(k^2\,\eta^2\,M\bigr)^{-2d}\, 4^{3\ell/2+\ell\delta/2} \gg \eta_\ell\, M^{\delta/3-d}, \end{aligned}$$

giving a contradiction when d is sufficiently small, and we are done.

Finally, we need to treat the case in which $h_\ell/k_\ell = h/k$ for some $\ell \in \{0,1,\ldots,L\}$. In this case, we still proceed as in the proof of Proposition 20 but we estimate the corresponding subsums $\sum_{n\in\mathbb{Z}} t(n)\, e(n\alpha)\, w_{\pm\ell}(n)$ in a manner similar to our application of Proposition 19, i.e. by further splitting them into shorter smooth subsums of lengths both $\gg \widetilde{U}$ and $\leqslant \widetilde{U}$ with an appropriate $\widetilde{U} = c\,U\,4^{-\ell}\,\bigl(k^2\,\eta^2\,M\bigr)^{-d}$ for a small constant $c \in \mathbb{R}_+$, and then estimating these shorter subsums using Theorem 18.

8.2 Voronoi summation formula and saddle-points: the main terms

Now we consider the smoothed sum (7). We start by estimating terms that arise when Voronoi summation formula is applied and Bessel functions are replaced by their asymptotic expressions. As always, the terms involving the K-Bessel function contribute a negligible amount. Asymptotics of the J-Bessel function lead to certain exponential integrals which are estimated by using standard tools. We will assume throughout the proof that $\eta \geqslant 0$, as the other case is similar.

The Voronoi summation formula says that

$$\begin{aligned} &\sum_{M_{-1}\leqslant n\leqslant M_2} t(n)\, e(n\alpha)\, w(n) \\ &= \frac{\pi i}{k\,\sinh \pi\kappa} \sum_{n=1}^{\infty} t(n)\, e\!\left(\frac{-n\overline{h}}{k}\right) \int_{M_{-1}}^{M_2} (J_{2i\kappa} - J_{-2i\kappa})\!\left(\frac{4\pi\sqrt{nx}}{k}\right) e(\eta x)\, w(x)\,\mathrm{d}x \\ &\qquad + \frac{4\cosh \pi\kappa}{k} \sum_{n=1}^{\infty} t(-n)\, e\!\left(\frac{n\overline{h}}{k}\right) \int_{M_{-1}}^{M_2} K_{2i\kappa}\!\left(\frac{4\pi\sqrt{nx}}{k}\right) e(\eta x)\, w(x)\,\mathrm{d}x. \end{aligned}$$

The sum involving K-Bessel function contributes $\ll 1$ as before. Replacing the difference between J-Bessel functions by the asymptotic expression (4) gives

$$\sum_{M_{-1}\leqslant n\leqslant M_2} t(n)\,e(n\alpha)\,w(n) = O(1) + \frac{C'}{k}\sum_{n=1}^{\infty} t(n)\,e\biggl(\frac{-n\overline{h}}{k}\biggr)$$

$$\cdot \int\limits_{M_{-1}}^{M_2} \frac{k^{1/2}}{n^{1/4}\,x^{1/4}} \sum_{\pm}(\pm 1)e\biggl(\pm\frac{2\sqrt{nx}}{k}\biggr)\,e\biggl(\mp\frac{1}{8}\biggr)\,g_{\pm}(x;n,k)\,e(\eta x)\,w(x)\,\mathrm{d}x,$$

where

$$g_{\pm}(x;n,k) = 1 + \sum_{\ell=1}^{K} c_{\ell}^{\pm}\,k^{\ell}\,n^{-\ell/2}\,x^{-\ell/2},$$

and $C' = i/\sqrt{2}$, just like in the proof of Theorem 18 as the error term from the J-Bessel asymptotics gives the contribution $\ll 1$ when K is fixed and chosen to be large enough. Now by Lemma 12 we have

$$\sum_{n=1}^{\infty} t(n)\,n^{-1/4} \int\limits_{M_{-1}}^{M_2} k^{-1/2}\,x^{-1/4}\,e\biggl(x\eta + \frac{2\sqrt{nx}}{k}\biggr)\,g_{+}(x;n,k)\,w(x)\,\mathrm{d}x \ll 1,$$

and so the main terms come from the integrals involving g_{-}.

Let c be a positive constant so that the estimate

$$\eta - \frac{\sqrt{n}}{k\sqrt{x}} \gg \frac{\sqrt{n}}{k\sqrt{M}}$$

holds when $n > cN$, where $N = k^2\,\eta^2\,M$ as before. A direct application of Lemma 12 gives

$$\sum_{n>cN} t(n)\,n^{-1/4} \int\limits_{M_{-1}}^{M_2} k^{-1/2}\,x^{-1/4}\,e\biggl(x\eta - \frac{2\sqrt{nx}}{k}\biggr)\,g_{-}(x;n,k)\,w(x)\,\mathrm{d}x \ll 1.$$

For the terms with $n \leqslant cN$ we split $g_{-}(x;n,k)$ into two parts 1 and $g_{-}(x;n,k) - 1$ and estimate corresponding terms differently.

For the first term, using the second derivative test we get

$$\int\limits_{M_{-1}}^{M_2} x^{-1/4}\,e\biggl(x\eta - \frac{2\sqrt{nx}}{k}\biggr)\,(g_{-}(x;n,k) - 1)\,w(x)\,\mathrm{d}x \ll \frac{k^{3/2}}{n^{3/4}}.$$

Therefore we have

$$\sum_{n\leqslant cN} t(n)\,n^{-1/4} \int\limits_{M_{-1}}^{M_2} k^{-1/2}\,x^{-1/4}\,e\biggl(x\eta - \frac{2\sqrt{nx}}{k}\biggr)\,(g_{-}(x;n,k) - 1)\,w(x)\,\mathrm{d}x$$

$$\ll k \sum_{n\leqslant cN} \frac{|t(n)|}{n} \ll k\,N^{\varepsilon} = k\,(k^2\,\eta^2\,M)^{\varepsilon}.$$

The remaining terms are treated using the first saddle point lemma, Theorem 10. For $1 \leqslant n < cN$ we get

$$\int_{M_{-1}}^{M_2} e\Big(x\eta - \frac{2\sqrt{nx}}{k}\Big)\, w(x)\, x^{-1/4}\,\mathrm{d}x = \xi(n)\cdot \frac{\sqrt{2}\, n^{1/4}}{\sqrt{k}\,\eta}\, e\Big(-\frac{n}{k^2\eta} + \frac{1}{8}\Big)$$
$$+ O\Big((M_2 - M_{-1})\,\big(1 + M^J U^{-J}\big)\, M^{-1/4}\, e^{-A|\eta| M - A\sqrt{nM}/k}\Big)$$
$$+ O\Big(\frac{k^{3/2}}{n^{3/4}} + \chi(n)\,\frac{M^{1/4}\,k}{\sqrt{n}}\Big)$$
$$+ O\left(M^{-1/4}\, U^{-J} \sum_{j=0}^{J} \left(\left|\eta - \frac{\sqrt{n}}{k\sqrt{M_{-1} + jU}}\right| + \frac{n^{1/4}}{\sqrt{k}\, M^{3/4}}\right)^{-J-1}\right)$$
$$+ O\left(M^{-1/4}\, U^{-J} \sum_{j=0}^{J} \left(\left|\eta - \frac{\sqrt{n}}{k\sqrt{M_{2} - jU}}\right| + \frac{n^{1/4}}{\sqrt{k}\, M^{3/4}}\right)^{-J-1}\right),$$

where we have written for simplicity $\xi(\cdot) = \xi_J(\cdot/(k^2\eta^2))$, and where

$$\begin{cases} \xi(n) = 0 \text{ and } \chi(n) = 0 & \text{if } n \leqslant N_{-1} \text{ or } n \geqslant N_2, \\ \xi(n) = 1 \text{ and } \chi(n) = 0 & \text{if } N \leqslant n \leqslant N_1, \\ \xi(n) \ll 1 \text{ and } \chi(n) = 1 & \text{otherwise.} \end{cases}$$

Furthermore, ξ' is piecewise continuously differentiable and we have $\xi'(\cdot) \ll (k^2\,\eta^2\,U)^{-1}$ where the derivative exists.

The main term on the right-hand side produces the total contribution

$$\frac{1}{k\eta} \sum_{N \leqslant n \leqslant N_1} t(n)\, e\Big(-\frac{n\overline{h}}{k} - \frac{n}{k^2\eta}\Big) + \frac{1}{k\eta} \sum_{N_{-1} \leqslant n < N} t(n)\,\xi(n)\, e\Big(-\frac{n\overline{h}}{k} - \frac{n}{k^2\eta}\Big)$$
$$+ \frac{1}{k\eta} \sum_{N_1 < n \leqslant N_2} t(n)\,\xi(n)\, e\Big(-\frac{n\overline{h}}{k} - \frac{n}{k^2\eta}\Big).$$

The first term is exactly what appears in the statement of the theorem. Let us first estimate the contribution of error terms arising from the saddle point lemma and after that estimate the contribution of the other main terms.

8.3 The error terms from the saddle point theorem

The first error term contributes

$$k^{-1/2} \sum_{1 \leqslant n \leqslant cN} \frac{|t(n)|}{n^{1/4}}\, (\Delta + U)\, \big(1 + M^J U^{-J}\big)\, M^{-1/4}$$
$$\cdot \exp\left(-A|\eta| M^{1/4} - \frac{A\sqrt{nM}}{k}\right)$$
$$\ll (\Delta + U)\, M^{J-1/4}\, N^{3/4} \cdot e^{-AM^{1/4}} \ll_A 1.$$

The second error term is also easy to handle:

$$k^{-1/2} \sum_{1 \leqslant n \leqslant cN} \frac{|t(n)|}{n^{1/4}} \left(\frac{k^{3/2}}{n^{3/4}} + \chi(n)\frac{M^{1/4} k}{n^{1/2}}\right)$$
$$\ll k\,(k^2\eta^2 M)^{\varepsilon + d + \vartheta + 1/4} \ll M^{1/2}\,(k^2\eta^2 M)^{d - 1/12 + \varepsilon}$$

just by using partial summation.

The estimation of the other two error terms is covered by the following lemma.

Lemma 23. *Let c be any given positive constant. Let $T \in [M_{-1}, M_2]$. Then*

$$k^{-1/2} \sum_{1 \leqslant n \leqslant cN} \frac{t(n)}{n^{1/4}} M^{-1/4} U^{-J} \left(\left| \eta - \frac{\sqrt{n}}{\sqrt{T}\, k} \right| + \frac{n^{1/4}}{\sqrt{k}\, M^{3/4}} \right)^{-J-1} \ll \sqrt{M}\, (k^2 \eta^2 M)^{\varepsilon + \vartheta + 1/2 - Jd}.$$

Proof. We estimate the left-hand side as $\ll S_1 + S_2 + S_3$, where

$$S_1 = k^{-1/2} M^{-1/4} U^{-J} \sum_{|n - k^2 \eta^2 T| \leqslant \sqrt{N}} n^{\vartheta + \varepsilon - 1/4} \left(\frac{n^{1/4}}{\sqrt{k}\, M^{3/4}} \right)^{-J-1},$$

$$S_2 = k^{-1/2} M^{-1/4} U^{-J} \sum_{1 \leqslant n \leqslant k^2 \eta^2 T - \sqrt{N}} |t(n)|\, n^{-1/4} \left| \eta - \frac{\sqrt{n}}{\sqrt{T}\, k} \right|^{-J-1},$$

and

$$S_3 = k^{-1/2} M^{-1/4} U^{-J} \sum_{k^2 \eta^2 T + \sqrt{N} \leqslant n \leqslant cN} |t(n)|\, n^{-1/4} \left| \eta - \frac{\sqrt{n}}{\sqrt{T}\, k} \right|^{-J-1}.$$

Next we compute the claimed upper bound for each of them. Observe that by partial summation

$$\begin{aligned} S_1 &= M^{1/2} (k^2 \eta^2 M)^{-Jd} k^{J/2} M^{J/4} \eta^{J/2} \sum_{|n - k^2 \eta^2 T| \leqslant \sqrt{N}} n^{\vartheta + \varepsilon - 1/2 - J/4} \\ &\ll M^{1/2} (k^2 \eta^2 M)^{-Jd} k^{J/2} M^{J/4} \eta^{J/2} (k^2 \eta^2 M)^{\vartheta + \varepsilon - J/4} \\ &\ll M^{1/2} (k^2 \eta^2 M)^{\varepsilon - Jd + \vartheta}. \end{aligned}$$

The sum S_2 is estimated as follows:

$$\begin{aligned} S_2 &\ll k^{-1/2} M^{-1/4} M^{-J/2} \eta^{J/2} N^{-dJ} \sum_{n \leqslant k^2 \eta^2 T - \sqrt{N}} \frac{|t(n)|}{n^{1/4}} \left| \eta - \frac{\sqrt{n}}{k \sqrt{T}} \right|^{-J-1} \\ &\ll k^{-1/2} M^{-1/4} M^{-J/2} \eta^{J/2} N^{-dJ} k^{J+1} M^{J/2 + 1/2} \\ &\quad \cdot \sum_{n \leqslant k^2 \eta^2 T - \sqrt{N}} \frac{|t(n)|}{n^{1/4}} \cdot \frac{\left| k \eta \sqrt{T} + \sqrt{n} \right|^{J+1}}{|k^2 \eta^2 T - n|^{J+1}} \\ &\ll k^{-1/2} M^{-1/4} M^{-J/2} \eta^{J/2} N^{-dJ} k^{J+1} M^{J/2 + 1/2} N^{-J/2 - 1/2} N^{J/2 + 1/2} N^{3/4} \\ &\ll k^{1/2 + J} \eta^{J/2} M^{1/4} N^{-dJ + 3/4} \ll M^{3/8} N^{3/4 - dJ} \ll M^{1/2} N^{1/2 - dJ}. \end{aligned}$$

Finally, the sum S_3 is estimated in the same manner as S_2.

8.4 Removing the weight function ξ

By partial summation it is enough to deal with the sum without $\xi(n)$. Observe that $N - N_{-1} = N_2 - N_1 = k^2 \eta^2 J U$ and

$$k^2 \eta^2 U = (k^2 \eta^2 M)^{d + 1/2} k \eta^{1/2} \ll (k^2 \eta^2 M)^{2/3} = N^{2/3}$$

for sufficiently small $d \in \mathbb{R}_+$.

Therefore, using Theorem 1 we get that the other two main terms given by the main term of the saddle point lemma contribute

$$
\begin{aligned}
&\frac{1}{k\eta}\left(\sum_{N_{-1}\leqslant n<N}+\sum_{N_1<n\leqslant N_2}\right) t(n)\,\xi(n)\,e\left(-\frac{n\overline{h}}{k}-\frac{n}{k^2\eta}\right)\\
&\ll \frac{1}{k\eta}\left(k^2\,\eta^2\,M^{1/2}\,\eta^{-1/2}\left(k^2\,\eta^2\,M\right)^d\right)^{1/6-\vartheta}\cdot\left(k^2\,\eta^2\,M\right)^{1/3+\vartheta+\varepsilon}\\
&\ll M^{1/2}\left(k^2\,\eta^2\,M\right)^{\vartheta/2-1/12+d/6-d\vartheta+\varepsilon}\left(k\,\eta^{1/2}\right)^{1/6-\vartheta}\\
&\ll M^{1/2}\left(k^2\,\eta^2\,M\right)^{\vartheta/2-1/12+d/6-d\vartheta+\varepsilon}
\end{aligned}
$$

for small enough $d\in\mathbb{R}_+$.

At this point we have proved that

$$
\begin{aligned}
&\sum_{M_{-1}\leqslant n\leqslant M_2} t(n)\,w(n)\,e(\alpha n)\\
&\qquad=\frac{1}{k\eta}\sum_{N\leqslant n\leqslant N_1} t(n)\,e(-\beta n)+O(M^{1/2}\,(k^2\eta^2M)^{\varepsilon+\vartheta+1/2-Jd})\\
&\qquad\qquad+O(M^{1/2}\,(k^2\,\eta^2\,M)^{\varepsilon+d-1/12+\vartheta/2})+O\left(k\,(k^2\,\eta^2\,M)^{\varepsilon}\right).
\end{aligned}
$$

Furthermore, using (8), this tells that

$$
\begin{aligned}
&\sum_{M\leqslant n\leqslant M+\Delta} t(n)\,e(\alpha n)\\
&\qquad=\frac{1}{k\eta}\sum_{N\leqslant n\leqslant N_1} t(n)\,e(-\beta n)+O\Big(M^{1/2}\,(k^2\eta^2M)^{\varepsilon+\vartheta+1/2-Jd}\Big)\\
&\qquad\qquad+O\Big(M^{1/2}\,(k^2\,\eta^2\,M)^{\varepsilon+d-1/12+\vartheta/2}\Big)+O\big(k\,(k^2\eta^2M)^{\varepsilon}\big).
\end{aligned}
$$

Choosing J sufficiently large depending on d, and letting $d\in\mathbb{R}_+$ to be arbitrarily small finishes the proof.

9 Proof of Theorem 7

We shall prove theorem 7 first near rational points, and then iterate the approximate functional equation in the remaining cases until either we end up near a rational point or the sum in question has become shorter than some given constant.

9.1 Logarithm removal near rational points

The following lemma, which we will soon prove, covers the logarithm removal near rational points.

Lemma 24. *Let $M\in[1,\infty[$, let $\alpha\in\mathbb{R}$, and $h\in\mathbb{Z}$ and $k\in\mathbb{Z}$ be coprime with $1\leqslant k\leqslant M^{1/4}$, and $\alpha=h/k+\eta$ with $|\eta|\leqslant k^{-1}\,M^{-1/4}$. If $k^2\,\eta^2\,M<1/2$, then*

$$\sum_{M\leqslant n\leqslant 2M} t(n)\,e(n\alpha)\ll k^{(1-6\vartheta)/(4-6\vartheta)}\,M^{3/(8-12\vartheta)+\varepsilon}.$$

In particular, with the exponent $\vartheta=7/64$ the upper bound is $\ll M^{203/428+\varepsilon}\ll M^{1/2}$.

The following Voronoi type identity for Maass forms can be found in Section 12 of Meurman's paper [Me2].

Theorem 25. *Let $x \in [1, \infty[$, and let h and k be coprime integers with $k \geqslant 1$. Then*

$$\sum_{n \leqslant x}{}' t(n)\, e\Big(\frac{nh}{k}\Big) = \frac{2\pi}{k \sinh \pi\kappa} \sum_{n=1}^{\infty} t(n)\, e\Big(\frac{-n\bar{h}}{k}\Big) \int_0^x \Re\bigg(i\, J_{2i\kappa}\Big(\frac{4\pi\sqrt{nv}}{k}\Big)\bigg)\, \mathrm{d}v$$
$$+ \frac{4\cosh \pi\kappa}{k} \sum_{n=1}^{\infty} t(-n)\, e\Big(\frac{n\bar{h}}{k}\Big) \int_0^x K_{2i\kappa}\Big(\frac{4\pi\sqrt{nv}}{k}\Big)\, \mathrm{d}v,$$

and where the series are boundedly convergent for x restricted in any bounded subinterval of $[1, \infty[$.

The integrals involving the J-Bessel function will have an asymptotic expansion reminiscent of those for the J-Bessel function itself. The following asymptotics for the J-Bessel function integral are obtained from Section 6 of [Me2], and the asymptotics for the integral involving the K-Bessel function is easily obtained from the asymptotic properties of $K_{2i\kappa}$.

Lemma 26. *Let $n \in \mathbb{Z}_+$, $x \in [1, \infty[$, and $k \in \mathbb{Z}_+$. If $n\,x \gg k^2$, then we have an asymptotic expansion*

$$\int_0^x \Re\bigg(i\, J_{2i\kappa}\Big(\frac{4\pi\sqrt{nv}}{k}\Big)\bigg)\, \mathrm{d}v$$
$$= k^{3/2}\, n^{-3/4}\, x^{1/4} \sum_{\pm} A_{1,\pm}\, e\Big(\pm\frac{2\sqrt{nx}}{k}\Big) + A_2\, k^2\, n^{-1}$$
$$+ k^{5/2}\, n^{-5/4}\, x^{-1/4} \sum_{\pm} A_{3,\pm}\, e\Big(\pm\frac{2\sqrt{nx}}{k}\Big) + O_\kappa(k^{7/2}\, n^{-7/4}\, x^{-3/4}),$$

where $A_{1,+}$, $A_{1,-}$, A_2, $A_{3,+}$ and $A_{3,-}$ are some constants only depending on κ, and the implicit constant in the lower bound $n\,x \gg k^2$. Similarly, we have the asymptotic expansion

$$\int_0^x K_{2i\kappa}\Big(\frac{4\pi\sqrt{nv}}{k}\Big)\, \mathrm{d}v = B_2\, k^2\, n^{-1} + O_{\kappa,C}(k^{2+C}\, n^{-1-C/2}\, x^{-C/2}),$$

where $C \in \mathbb{R}_+$ is arbitrary and B_2 is a constant only depending on κ and the implicit constant in $n\,x \gg k^2$.

We need one more lemma before the proof as the special value $L(1, h/k)$ will appear there.

Lemma 27. *Let h and k be coprime integers with $k \geqslant 1$. Then*

$$\sum_{n=1}^{\infty} \frac{t(n)}{n}\, e\Big(\frac{nh}{k}\Big) \ll k^{\varepsilon}.$$

Proof. It is proved in [Me2] that the rationally additively twisted L-function attached to our fixed Maass form,

$$L\Big(s, \frac{h}{k}\Big) = \sum_{n=1}^{\infty} \frac{t(n)}{n^s}\, e\Big(\frac{nh}{k}\Big),$$

at first defined only for complex numbers s with $\Re s > 1$, has an entire analytic extension to $\mathbb{C}$. Furthermore, this L-function, in a sense, satisfies a functional equation with Γ-factors. Using the fact that the twisted L-functions are $\ll_\delta 1$ on the vertical line $\Re s = 1+\delta$ for any fixed $\delta \in \mathbb{R}_+$, the functional equations combined with Stirling's formula easily give the bound

$$L\left(s, \frac{h}{k}\right) \ll_\delta k^{1+2\delta}\,(1+|t|)^{1+2\delta} \quad \text{on the vertical line} \quad \Re s = -\delta.$$

Phragmén–Lindelöf principle then tells us that

$$L\left(s, \frac{h}{k}\right) \ll_\delta k^{1+\delta-\sigma}\,(1+|t|)^{1+\delta-\sigma}$$

in the vertical strip $-\delta \leqslant \Re s \leqslant 1+\delta$. Applying this with $s = 1$ gives the Lemma. For more details about the functional equations used here, we refer to Section 2 of [Me2].

Proof of Lemma 24. We begin by integrating by parts:

$$\sum_{M\leqslant n\leqslant 2M} t(n)\,e(n\alpha) = e(\eta x)\sum_{n\leqslant x} t(n)\,e\left(\frac{nh}{k}\right)\Bigg]_{x=M}^{x=2M} - 2\pi i\eta \int_M^{2M} e(\eta x)\sum_{n\leqslant x} t(n)\,e\left(\frac{nh}{k}\right)\mathrm{d}x.$$

As $k \leqslant M^{1/4}$, Theorem 4 immediately tells us that the substitution terms are $\ll k^{(1-6\vartheta)/(4-6\vartheta)}\,M^{3/(8-12\vartheta)+\varepsilon}$. We will prove that the term involving the integral is actually $\ll k^{1/2}\,M^{1/4}$. The full Voronoi identity for Maass forms tells us that

$$\begin{aligned}
&\eta\int_M^{2M} e(\eta x)\sum_{n\leqslant x} t(n)\,e\left(\frac{nh}{k}\right)\mathrm{d}x\\
&= \frac{2\pi\,\eta}{k\,\sinh\pi\kappa}\sum_{n=1}^{\infty} t(n)\,e\left(\frac{-n\overline{h}}{k}\right)\int_M^{2M} e(\eta x)\int_0^x \Re\left(i\,J_{2i\kappa}\left(\frac{4\pi\sqrt{nv}}{k}\right)\right)\mathrm{d}v\,\mathrm{d}x\\
&\quad + \frac{4\eta\,\cosh\pi\kappa}{k}\sum_{n=1}^{\infty} t(-n)\,e\left(\frac{n\overline{h}}{k}\right)\int_M^{2M} e(\eta x)\int_0^x K_{2i\kappa}\left(\frac{4\pi\sqrt{nv}}{k}\right)\mathrm{d}v\,\mathrm{d}x.
\end{aligned}$$

We emphasize that termwise integration of the series is allowed since the series converge boundedly. We note that the integral $\int_0^x$ involving the K-Bessel function has better asymptotic behaviour than the similar integral involving the J-Bessel function, and since the two series otherwise have largely the same shape, it is enough to consider the series involving $J_{2i\kappa}$.

Next we simply replace the $\int_0^x \Re(i\ldots)\mathrm{d}v$ by the asymptotics given by Lemma 26. We start with the contribution from either of the first main terms. Since $k^2\,\eta^2\,M < 1/2$, we have

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\pm\frac{2\sqrt{nx}}{k}+\eta\right) = \pm\frac{\sqrt{n}}{k\,\sqrt{x}}+\eta \asymp \frac{\sqrt{n}}{k\,\sqrt{x}} \asymp n^{1/2}\,k^{-1}\,M^{-1/2}.$$

Thus, using the first derivative test, the contribution from these terms is

$$\ll \frac{\eta}{k}\sum_{n=1}^{\infty}|t(n)|\,\frac{k^2}{n}\left|\int_M^{2M} e(\eta x)\,\frac{n^{1/4}\,x^{1/4}}{k^{1/2}}\,e\Big(\pm\frac{2\sqrt{nx}}{k}\Big)\,\mathrm{d}x\right|$$

$$\ll \frac{1}{k^2\,M^{1/2}}\sum_{n=1}^{\infty}|t(n)|\,\frac{k^2}{n}\cdot\frac{n^{1/4}\,M^{1/4}}{k^{1/2}}\cdot\frac{k\,M^{1/2}}{n^{1/2}} \ll k^{1/2}\,M^{1/4}.$$

The contribution from the constant term of the asymptotics is

$$\ll \frac{\eta}{k}\sum_{n=1}^{\infty}t(n)\,e\Big(\frac{-n\overline{h}}{k}\Big)\,\frac{k^2}{n}\int_M^{2M} e(\eta t)\,\mathrm{d}t$$

$$\ll k\sum_{n=1}^{\infty}\frac{t(n)}{n}\,e\Big(\frac{-n\overline{h}}{k}\Big) \ll k^{1+\varepsilon} \ll k^{1/2}\,M^{1/8+\varepsilon}.$$

The contribution from the third main terms is clearly smaller than that from the first main terms since $k\,n^{-1/2}\,x^{-1/2} \ll 1$. Finally, the contribution from the O-term of the asymptotics contributes

$$\ll \frac{\eta}{k}\sum_{n=1}^{\infty}|t(n)|\,k^{7/2}\,n^{-7/4}\int_M^{2M} x^{-3/4}\,\mathrm{d}x$$

$$\ll k^{-1}\,M^{-1/2}\,k^{5/2}\,M^{1/4} = k^{3/2}\,M^{-1/4} \ll k^{1/2},$$

and we are done.

9.2 Away from rational points; applying the approximate functional equation

When $k^2\,\eta^2\,M \gg 1$, the logarithm removal is implemented quite easily using the approximate functional equation. The result will be as follows:

Lemma 28. *Let $M \in [1,\infty[$, let $\alpha \in \mathbb{R}$, let h and k be coprime integers with $1 \leqslant k \leqslant M^{1/4}$, and let $\alpha = h/k + \eta$ with $|\eta| \leqslant k^{-1}\,M^{-1/4}$. If $k^2\,\eta^2\,M \gg 1$, then*

$$\sum_{M\leqslant n\leqslant 2M} t(n)\,e(n\alpha) \ll M^{1/2}.$$

We start by applying the approximate functional equation, obtaining:

$$\frac{1}{M^{1/2}}\sum_{M\leqslant n\leqslant 2M} t(n)\,e(n\alpha) = \frac{1}{(k^2\,\eta^2\,M)^{1/2}}\sum_{k^2\eta^2 M\leqslant n\leqslant 2k^2\eta^2 M} t(n)\,e(n\beta) + O\big((k^2\,\eta^2\,M)^{\vartheta/2-1/12+\varepsilon}\big),$$

where $\beta = -\overline{h}/k - (k^2\,\eta)^{-1}$. Write for β a rational approximation $\beta = h_1/k_1 + \eta_1$ with h_1 and k_1 coprime and $1 \leqslant k_1 \leqslant (k^2\,\eta^2\,M)^{1/4}$ and with remainder satisfying $|\eta_1| \leqslant k_1^{-1}\,(k^2\,\eta^2\,M)^{-1/4}$. If $k_1^2\,\eta_1^2\,(k^2\,\eta^2\,M) < 1/2$, then the first term on the right-hand side is $\ll 1$ by Lemma 24, and the error term is clearly $\ll 1$, and we are done.

If instead $k_1^2\,\eta_1^2\,(k^2\,\eta^2\,M) \gg 1$, then we apply the approximate functional equation again to the right-hand side, and iterate the above argument as many times as necessary. Since the length of the new sum from the approximate functional equation is at most the square root of the length of the previous sum, the exponential sum term will eventually be covered by Lemma 24 or become shorter than some constant length. In either case, the sum will ultimately be $\ll 1$, and the error terms will form a nice geometric progression which sums up to $\ll 1$.

10 Proof of Theorem 8

Let us begin with a simple corollary to Theorems 6 and 1.

Lemma 29. *Let $M \in [1,\infty[$ and $\Delta \in [1,M]$ with $M \gg 1$ and $M^{2/3} \ll \Delta \ll M^{3/4}$. Furthermore, let $\alpha \in \mathbb{R}$, $h \in \mathbb{Z}$, $k \in \mathbb{Z}_+$ and $\eta \in \mathbb{R}$ with h and k coprime, $\alpha = h/k+\eta$, $k \leqslant M^{1/4}$ and $|\eta| \leqslant k^{-1} M^{-1/4}$. Also, let $k^2\,\eta^2\,M = M^\gamma$ with $\gamma \in \mathbb{R}_+$. Then*

$$\sum_{M \leqslant n \leqslant M+\Delta} t(n)\,e(n\alpha) \ll \Delta^{1/6-\vartheta}\,M^{1/3+\vartheta+\varepsilon} + M^{1/2-(1/12-\vartheta/2)\gamma+\varepsilon}.$$

Proof. The approximate functional equation of Theorem 6 immediately gives

$$\sum_{M \leqslant n \leqslant M+\Delta} t(n)\,e(n\alpha) = \frac{1}{k\eta} \sum_{k^2\eta^2 M \leqslant n \leqslant k^2\eta^2(M+\Delta)} t(n)\,e(n\beta) + O(M^{1/2-(1/12-\vartheta/2)\gamma+\varepsilon}),$$

where $\beta = -\overline{h}/k - 1/(k^2\,\eta)$. We also have $k^2\,\eta^2\,\Delta \ll \left(k^2\,\eta^2\,M\right)^{2/3}$ since this is equivalent to $\left(k^2\,\eta^2\right)^{1/3}\Delta \ll M^{2/3}$, which is true in view of

$$\left(k^2\,\eta^2\right)^{1/3}\Delta \ll M^{-1/6}\,M^{3/4} = M^{7/12} \ll M^{2/3}.$$

Thus, we may use Theorem 1 to estimate

$$\sum_{k^2\eta^2 M \leqslant n \leqslant k^2\eta^2(M+\Delta)} t(n)\,e(n\beta) \ll \left(k^2\,\eta^2\,\Delta\right)^{1/6-\vartheta}\,\left(k^2\,\eta^2\,M\right)^{1/3+\vartheta+\varepsilon} \ll k\,\eta\,\Delta^{1/6-\vartheta}\,M^{1/3+\vartheta+\varepsilon},$$

and the lemma has been proved.

Next, we shall prove from the Voronoi summation formula another estimate for the same sum which works nicely for a different range of $k^2\,\eta^2\,M$.

Lemma 30. *Let $M, \Delta \in \mathbb{R}_+$ with $M \gg 1$ and $M^\beta \ll \Delta \ll M^{3/4}$, where $\beta \in [2/3, 3/4]$. Furthermore, let $\alpha \in \mathbb{R}$, $h, k \in \mathbb{Z}$ and $\eta \in \mathbb{R}$ with h and k coprime, $1 \leqslant k \leqslant M^{1/4}$, $\alpha = h/k+\eta$ and $|\eta| \leqslant k^{-1}\,M^{-1/4}$. Also, let $w \in C_c^\infty(\mathbb{R}_+)$ be supported on $[M, M+\Delta]$, take values only from $[0,1]$ and satisfy $w^{(\nu)}(x) \ll_\nu \Delta^{-\nu}$ for all $x \in \mathbb{R}_+$ and $\nu \in \{0\} \cup \mathbb{Z}_+$. If $k^2\,\eta^2\,M \leqslant 1/2$, then*

$$\sum_{M \leqslant n \leqslant M+\Delta} t(n)\,e(n\alpha)\,w(n) \ll M^{9/8-\beta}.$$

If $k^2\,\eta^2\,M \gg 1$, then, for any $S \in [1,\infty[$,

$$\sum_{M \leqslant n \leqslant M+\Delta} t(n)\,e(n\alpha)\,w(n) \ll k^3\,|\eta|^{3/2}\,M^{3/2}\,\Delta^{-1}\,\left(k^2\,\eta^2\,M\right)^{\vartheta+\varepsilon}\,S^{-1} + k^{-1/2}\,\Delta\,M^{-1/4}\,\left(S + k^2\,\eta^2\,\Delta\right)\left(k^2\,\eta^2\,M\right)^{-1/4+\vartheta+\varepsilon} + M^{9/8-\beta}.$$

Proof. We shall apply the full Voronoi summation formula for one last time and write as before

$$\sum_{M\leqslant n\leqslant M+\Delta} t(n)\,e(n\alpha)\,w(n)$$

$$= \frac{\pi i}{k\sinh\pi\kappa}\sum_{n=1}^{\infty} t(n)\,e\biggl(\frac{-n\overline{h}}{k}\biggr)\int\limits_{M}^{M+\Delta}(J_{2i\kappa}-J_{-2i\kappa})\biggl(\frac{4\pi\sqrt{nx}}{k}\biggr)\,e(\eta x)\,w(x)\,\mathrm{d}x$$

$$+\ \frac{4\cosh\pi\kappa}{k}\sum_{n=1}^{\infty} t(-n)\,e\biggl(\frac{n\overline{h}}{k}\biggr)\int\limits_{M}^{M+\Delta}K_{2i\kappa}\biggl(\frac{4\pi\sqrt{nx}}{k}\biggr)\,e(\eta x)\,w(x)\,\mathrm{d}x.$$

As in the proof of Theorem 18, the K-terms give only the small contribution $\ll 1$. Also, applying the asymptotics (4) with a sufficiently large fixed $K\in\mathbb{Z}_+$, the error terms also give $\ll 1$. Thus, we have

$$\sum_{M\leqslant n\leqslant M+\Delta} t(n)\,e(n\alpha)\,w(n) = k^{-1/2}\sum_{n=1}^{\infty} t(n)\,n^{-1/4}\,e\biggl(\frac{-n\overline{h}}{k}\biggr)$$

$$\cdot\int\limits_{M}^{M+\Delta}x^{-1/4}\sum_{\pm}A_{\pm}\,e\biggl(\pm\frac{2\sqrt{nx}}{k}\biggr)\,e\biggl(\pm\frac{1}{8}\biggr)\,g_{\pm}(x;n,k)\,e(\eta x)\,w(x)\,\mathrm{d}x+O(1),$$

where $A_{\pm}$ are constants, and

$$g_{\pm}(x;n,k)=1+\sum_{\ell=1}^{K}c_{\ell}^{\pm}\,k^{\ell}\,n^{-\ell/2}\,x^{-\ell/2}.$$

If $k^2\,\eta^2\,M\leqslant 1/2$, then we may simply estimate using Lemma 12 with $P=2$ that the infinite series is

$$\ll k^{-1/2}\sum_{n=1}^{\infty}|t(n)|\,n^{-1/4}\,M^{-1/4}\left(\Delta\,n^{1/2}\,k^{-1}\,M^{-1/2}\right)^{-2}\Delta$$

$$\ll k^{3/2}\,M^{3/4}\,\Delta^{-1}\ll M^{3/8}\,M^{-\beta}\,M^{3/4}\ll M^{9/8-\beta}.$$

Thus, we may focus on the case $k^2\,\eta^2\,M\gg 1$.

Next, writing $X=k^2\,\eta^2\,M$, the high-frequency terms $n>2X$ contribute, again using Lemma 12 with $P=2$,

$$\ll k^{-1/2}\sum_{n>2X}|t(n)|\,n^{-1/4}\,M^{-1/4}\left(\Delta\,n^{1/2}\,k^{-1}\,M^{-1/2}\right)^{-2}\Delta$$

$$\ll k^{3/2}\,M^{3/4}\,\Delta^{-1}\,X^{-1/4}\ll M^{3/8}\,M^{3/4}\,M^{-\beta}\,M^{-\gamma/4}\ll M^{9/8-\beta-\gamma/4}.$$

And so we are left with

$$\sum_{M\leqslant n\leqslant M+\Delta} t(n)\,e(n\alpha)\,w(n) = k^{-1/2}\sum_{\pm}A_{\pm}\sum_{n\leqslant 2X}t(n)\,n^{-1/4}\,e\biggl(\frac{-n\overline{h}}{k}\biggr)$$

$$\cdot\int\limits_{M}^{M+\Delta}x^{-1/4}\,g_{\pm}(x;n,k)\,e\biggl(\pm\frac{2\sqrt{nx}}{k}+\eta x\biggr)\,w(x)\,\mathrm{d}x+O(M^{9/8-\beta-\gamma/4})+O(1).$$

We shall split the sum $\sum_{n\leqslant 2X}$ into three parts

$$\sum_{n\leqslant 2X}=\sum_{n<X-S}+\sum_{X-S\leqslant n\leqslant X'+S}+\sum_{X'+S<n\leqslant 2X},$$

where for simplicity $X' = k^2\,\eta^2\,(M+\Delta)$ and S is a parameter to be chosen later but which satisfies $S \gg 1$. The first and third sums might be empty; this happens when $S \gg k^2\,\eta^2\,M$. Also, large values of S pose no problems in the middle terms as they are estimated via absolute values. Let us first consider the case $S \ll k^2\,\eta^2\,M$.

The third sum is estimated using Lemma 12 with $P = 2$ to get

$$\begin{aligned}
&k^{-1/2} \sum_{X'+S<n\leqslant 2X} \cdots \\
&\ll k^{-1/2} \sum_{X'+S<n\leqslant 2X} |t(n)|\, n^{-1/4} \cdot M^{-1/4}\, \Delta^{-P} \left(\frac{\sqrt{n}}{k\sqrt{M+\Delta}} - |\eta| \right)^{-P} \Delta \\
&\ll k^{-1/2}\, M^{-1/4}\, \Delta^{-1}\, X^{\vartheta-1/4+\varepsilon}\, k^2\, M \sum_{X'+S<n\leqslant 2X} \frac{1}{(\sqrt{n}-\sqrt{X'})^2} \\
&\ll k^{3/2}\, M^{3/4}\, \Delta^{-1}\, X^{\vartheta-1/4+\varepsilon} \sum_{X'+S<n\leqslant 2X} \frac{X}{(n-X')^2} \\
&\ll k^{3/2}\, M^{3/4}\, \Delta^{-1}\, X^{\vartheta+3/4+\varepsilon}\, S^{-1} \ll k^3\, |\eta|^{3/2}\, M^{3/2}\, \Delta^{-1}\, X^{\vartheta+\varepsilon}\, S^{-1}.
\end{aligned}$$

The first terms are estimated similarly, but with a dyadic split over the range of n:

$$\begin{aligned}
&k^{-1/2} \sum_{n<X-S} \cdots \\
&\ll k^{-1/2} \sum_{n<X-S} |t(n)|\, n^{-1/4} \cdot M^{-1/4}\, \Delta^{-P} \left(\frac{\sqrt{n}}{k\sqrt{M}} - |\eta| \right)^{-P} \Delta \\
&\ll k^{-1/2}\, M^{-1/4}\, \Delta^{-1}\, k^2\, M \sum_{\substack{L\leqslant X-S \\ \text{dyadic}}} \sum_{L\leqslant n<2L} \frac{|t(n)|}{n^{1/4} \left|\sqrt{n}-\sqrt{X}\right|^2} \\
&\ll k^{3/2}\, M^{3/4}\, \Delta^{-1}\, X \sum_{\substack{L\leqslant X-S \\ \text{dyadic}}} L^{\vartheta-1/4+\varepsilon} \sum_{L\leqslant n<2L} \frac{1}{|n-X|^2} \\
&\ll k^{3/2}\, M^{3/4}\, \Delta^{-1}\, X^{\vartheta+3/4+\varepsilon}\, S^{-1} \ll k^3\, |\eta|^{3/2}\, M^{3/2}\, \Delta^{-1}\, X^{\vartheta+\varepsilon}\, S^{-1}.
\end{aligned}$$

The middle terms are estimated by absolute values to get

$$\begin{aligned}
k^{-1/2} \sum_{X-S\leqslant n\leqslant X'+S} \cdots &\ll k^{-1/2} \sum_{X-S\leqslant n\leqslant X'+S} |t(n)|\, n^{-1/4} \cdot \Delta\, M^{-1/4} \\
&\ll k^{-1/2}\, \Delta\, M^{-1/4} \left(S + k^2\, \eta^2\, \Delta \right) X^{-1/4+\vartheta+\varepsilon}.
\end{aligned}$$

When $S \leqslant X/2$, say, then the last estimate follows immediately from estimate by absolute values. If $S > X/2$, we may use a dyadic split to estimate:

$$\begin{aligned}
\sum_{X-S\leqslant n\leqslant X'+S} \frac{|t(n)|}{n^{1/4}} &\ll \sum_{\substack{X-S\ll L\ll X'+S \\ \text{dyadic}}} \sum_{L<n\leqslant 2L} \frac{|t(n)|}{n^{1/4}} \\
&\ll \sum_{\substack{X-S\ll L\ll X'+S \\ \text{dyadic}}} L^{\vartheta+3/4+\varepsilon} \ll X^{\vartheta+3/4+\varepsilon} \ll S\, X^{\vartheta-1/4+\varepsilon}.
\end{aligned}$$

Finally, if $S \gg k^2\,\eta^2\,M$, then the first terms and third terms do not exist, and the middle terms satisfy the same upper bound as before.

We have obtained

$$\sum_{M\leqslant n\leqslant M+\Delta} t(n)\,e(n\alpha)\,w(n) \ll k^3\,|\eta|^{3/2}\,M^{3/2}\,\Delta^{-1}\,X^{\vartheta+\varepsilon}\,S^{-1}$$
$$+ k^{-1/2}\,\Delta\,M^{-1/4}\left(S+k^2\,\eta^2\,\Delta\right)X^{-1/4+\vartheta+\varepsilon} + M^{9/8-\beta-\gamma/4}+1.$$

Lemma 31. *Let $M\in[1,\infty[$ and $\Delta\in[1,M]$ with $M\gg 1$ and $M^{2/3}\ll\Delta\ll M^{3/4}$, and let $\alpha\in\mathbb{R}$, $h\in\mathbb{Z}$, $k\in\mathbb{Z}_+$ and $\eta\in\mathbb{R}$ with $\alpha=h/k+\eta$, $(h,k)=1$, $k\leqslant M^{1/4}$ and $|\eta|\leqslant k^{-1}\,M^{-1/4}$. If $k^2\,\eta^2\,M\leqslant 1/2$, then*

$$\sum_{M\leqslant n\leqslant M+\Delta} t(n)\,e(n\alpha) \ll M^{11/24}+M^{4/9+\vartheta/3+\varepsilon}.$$

If $k^2\,\eta^2\,M\gg 1$, then

$$\sum_{M\leqslant n\leqslant M+\Delta} t(n)\,e(n\alpha) \ll M^{11/24}+M^{4/9+\vartheta/3+\varepsilon}+M^{3/8+\gamma/4+\gamma\vartheta+\varepsilon}+\Delta\,M^{-1/4+\mu+\varepsilon},$$

provided that $\Delta\ll M^{1+\mu-3\gamma/4-\gamma\vartheta}$, where $\gamma\in\mathbb{R}$ is such that $k^2\,\eta^2\,M=M^{\gamma}$, and $\mu\in[0,\infty[$.

Proof. Let us use the same weight functions w_ℓ as in the proof of Theorem 1 (cf. also Fig. 4), and let us pick $L\in\mathbb{Z}_+$ so that $\Delta\,4^{-L}\asymp M^{2/3}$. Theorem 1 says that

$$\sum_{M\leqslant n\leqslant M+\Delta} t(n)\,e(n\alpha)\left(1-\sum_{\ell=-L}^{L} w_\ell(n)\right)$$
$$\ll \left(M^{2/3}\right)^{1/6-\vartheta} M^{1/3+\vartheta+\varepsilon} \ll M^{4/9+\vartheta/3+\varepsilon}.$$

Next, let us consider a single value $\ell\in\{-L,-L+1,\ldots,L\}$. If $k^2\,\eta^2\,M\leqslant 1/2$, then the previous lemma gives the upper bound

$$\sum_{n=-L}^{L} t(n)\,e(n\alpha)\,w_\ell(n) \ll M^{9/8-2/3} \ll M^{11/24},$$

so assume that $k^2\,\eta^2\,M\gg 1$.

If $k^2\,|\eta|\,M\,\Delta^{-1}\gg k^2\,\eta^2\,\Delta$, then we apply the previous lemma with the choice $S=k^2\,|\eta|\,M\,\Delta^{-1}$ which has been optimized so that the first two terms in the upper bound coincide. We get

$$\sum_{n\in\mathbb{Z}} t(n)\,e(n\alpha)\,w_\ell(n)$$
$$\ll k^{-1/2}\,\Delta\,M^{-1/4}\,k^2\,|\eta|\,M\,\Delta^{-1}\left(k^2\,\eta^2\,M\right)^{-1/4+\vartheta+\varepsilon}+M^{9/8-2/3}.$$

Writing $|\eta|=M^{(\gamma-1)/2}\,k^{-1}$, this is

$$\sum_{n\in\mathbb{Z}} t(n)\,e(n\alpha)\,w_\ell(n) \ll k^{1/2}\,M^{1/4}\left(k^2\,\eta^2\,M\right)^{1/4+\vartheta+\varepsilon}+M^{11/24}$$
$$\ll M^{3/8+\gamma/4+\gamma\vartheta+\varepsilon}+M^{11/24}.$$

If $k^2\,|\eta|\,M\,\Delta^{-1} \ll k^2\,\eta^2\,\Delta$, then we apply the previous lemma with $S = k^2\,\eta^2\,\Delta$ and get

$$\sum_{n\in\mathbb{Z}} t(n)\,e(n\alpha)\,w_\ell(n) \ll k\,|\eta|^{-1/2}\,\Delta^{-2}\,M^{3/2}\left(k^2\,\eta^2\,M\right)^{\vartheta+\varepsilon} + k^{3/2}\,\eta^2\,M^{-1/4}\,\Delta^2\left(k^2\,\eta^2\,M\right)^{\vartheta-1/4+\varepsilon} + M^{9/8-2/3}.$$

Now $M \ll |\eta|\,\Delta^2$, so that the first term is

$$\ll k\,|\eta|^{3/2}\,\Delta^2\,M^{-1/2}\left(k^2\,\eta^2\,M\right)^{\vartheta+\varepsilon},$$

and the second term is, writing again $|\eta| = M^{(\gamma-1)/2}\,k^{-1}$,

$$= k\,|\eta|^{3/2}\,\Delta^2\,M^{-1/2}\left(k^2\,\eta^2\,M\right)^{\vartheta+\varepsilon} \ll k^{-1/2}\,\Delta^2\,M^{3\gamma/4-5/4+\gamma\vartheta+\varepsilon}.$$

This is $\ll \Delta\,M^{-1/4+\mu+\varepsilon}$, provided that $\Delta \ll M^{1+\mu-3\gamma/4-\gamma\vartheta}$, and we are done.

Proof of Theorem 8. We shall get the result by combining Lemmas 29 and 31. To optimize the terms involving γ, we choose γ_0 so that

$$\frac{1}{2} - \left(\frac{1}{12} - \frac{\vartheta}{2}\right)\gamma_0 = \frac{3}{8} + \frac{\gamma_0}{4} + \gamma_0\vartheta, \quad \text{i.e.} \quad \gamma_0 = \frac{3}{12\vartheta+8}.$$

So, let $\alpha = h/k + \eta$ with h and k coprime integers, $1 \leqslant k \leqslant M^{1/4}$, and $\eta \in \mathbb{R}$ with $|\eta| \leqslant k^{-1}\,M^{-1/4}$. When $k^2\,\eta^2\,M \gg M^{\gamma_0}$, Lemma 29 gives

$$\sum_{M\leqslant n\leqslant M+\Delta} t(n)\,e(n\alpha) \ll \Delta^{1/6-\vartheta}\,M^{1/3+\vartheta+\varepsilon} + M^{3/8+(3+12\vartheta)/(32+48\vartheta)+\varepsilon}.$$

When $\Delta \ll M^{1+\mu-(3/4+\vartheta)\gamma_0}$ and $k^2\,\eta^2\,M \ll M^{\gamma_0}$, Lemma 31 gives

$$\sum_{M\leqslant n\leqslant M+\Delta} t(n)\,e(n\alpha) \ll M^{11/24} + M^{4/9+\vartheta/3+\varepsilon} + M^{3/8+(3+12\vartheta)/(32+48\vartheta)+\varepsilon} + \Delta\,M^{-1/4+\mu+\varepsilon},$$

where the exponent $\mu \in [0,\infty[$ is to be chosen later. It is easy to check that

$$\max\left\{\frac{11}{24}, \frac{4}{9}+\frac{\vartheta}{3}\right\} \leqslant \frac{3}{8} + \frac{3+12\vartheta}{32+48\vartheta} \quad \text{for} \quad \vartheta \in \left[0, \frac{7}{64}\right].$$

Also, it is easy to check that for $\Delta \ll M^{1+\mu-(3/4+\vartheta)\gamma_0}$, we have

$$\Delta^{1/6-\vartheta}\,M^{1/3+\vartheta+\varepsilon} \ll M^{3/8+(3+12\vartheta)/(32+48\vartheta)+\varepsilon} \quad \text{for} \quad \vartheta \in \left[0, \frac{7}{64}\right],$$

provided that $\mu \leqslant (3+12\vartheta)/(32+48\vartheta)$. Combining the above facts gives the estimate

$$\sum_{M\leqslant n\leqslant M+\Delta} t(n)\,e(n\alpha) \ll M^{3/8+(3+12\vartheta)/(32+48\vartheta)+\varepsilon} + \Delta\,M^{-1/4+\mu+\varepsilon},$$

for $\Delta \ll M^{1+\mu-(3/4+\vartheta)\gamma_0}$. However, when $\Delta \asymp M^{1+\mu-(3/4+\vartheta)\gamma_0}$, it is easy to check that

$$M^{3/8+(3+12\vartheta)/(32+48\vartheta)+\varepsilon} \asymp \Delta\,M^{-1/4+\mu+\varepsilon}, \quad \text{for any fixed} \quad \vartheta \in \left[0, \frac{7}{64}\right],$$

with equal exponents ε, of course, if we choose $\mu = 3\vartheta/(32+48\vartheta)$. This choice trivially satisfies the required upper bound $\leqslant (3+12\vartheta)/(32+48\vartheta)$. Thus, by splitting longer sums into sums of length $M^{1+\mu-(3/4+\vartheta)\gamma_0}$, and estimating these subsums separately, we have

$$\sum_{M\leqslant n\leqslant M+\Delta} t(n)\, e(n\alpha) \ll M^{3/8+(3+12\vartheta)/(32+48\vartheta)+\varepsilon} + \Delta\, M^{-1/4+\mu+\varepsilon},$$

with the weaker condition $\Delta \ll M^{3/4}$.

11 Ω-results from second moments

We end with some details about the Ω-results. Two important results are actually mean square results for which the key is the following truncated Voronoi identity due to Meurman [Me2].

Theorem 32. *Let $x \in [1,\infty[$, let $N \in \mathbb{R}_+$ be such that $N \ll x$, let k be a positive integer with $k \ll \sqrt{x}$, and let h be an integer coprime to k. Then*

$$\sum_{n\leqslant x} t(n)\, e\Big(\frac{nh}{k}\Big) = \frac{1}{\pi\sqrt{2}}\, k^{1/2}\, x^{1/4} \sum_{n\leqslant N} t(n)\, e\Big(\frac{-n\overline{h}}{k}\Big)\, n^{-3/4} \cdot \cos\Big(\frac{4\pi\sqrt{nx}}{k} - \frac{\pi}{4}\Big) + O\Big(k\, x^{1/2+\vartheta+\varepsilon}\, N^{-1/2}\Big).$$

This is simpler than the formulation in [Me2] where care was taken to retain an explicit dependence on ψ. For a fixed ψ, the above formulation follows easily. This simplified formulation can be used in the same manner as the truncated Voronoi identities for the error terms of the Dirichlet divisor problem or circle problem, or for holomorphic cusp forms.

For long exponential sums with rational additive twists, we have the following result, which is a Maass form analogue of Theorem 1.2 in [Ju4].

Theorem 33. *For $M \in [1,\infty[$, and for coprime integers h and k for which $k \geqslant 1$, we have*

$$\int_1^M \bigg|\sum_{n\leqslant x} t(n)\, e\Big(\frac{nh}{k}\Big)\bigg|^2 \mathrm{d}x = A\, k\, M^{3/2} + O(k^2\, M^{1+2\vartheta+\varepsilon}) + O(k^{3/2}\, M^{5/4+\vartheta+\varepsilon}),$$

where $A = (6\pi^2)^{-2} \sum_{n=1}^{\infty} |t(n)|^2\, n^{-3/2}$.

For sufficiently small k, the main term dominates:

Corollary 34. *For $M \in [1,\infty[$ with $M \gg 1$ and for coprime integers h and k for which $1 \leqslant k \ll M^{1/2-2\vartheta-\varepsilon}$, we have*

$$\int_1^M \bigg|\sum_{n\leqslant x} t(n)\, e\Big(\frac{nh}{k}\Big)\bigg|^2 \mathrm{d}x \asymp k\, M^{3/2}.$$

In particular, these sums are $\Omega(k^{1/2}\, M^{1/4})$.

Theorem 33 can be proved in the same way as Theorem 1.2 of [Ju4]. After a dyadic split of the range of integration, the integrals $\int_X^{2X}$ with $k \gg \sqrt{X}$, can be estimated simply by absolute values using the fact that the sum in the integrand is $\ll x^{1/2}$ by Theorem 7. In the other subintegrals, we proceed exactly as in [Ju4], with

Theorem 1.1 of [Ju4] being replaced by the above Theorem 32. Principally the extra terms involving ϑ come straight from the error term of Theorem 32. There is one other place where ϑ appears, in the estimation of the sums S_1 and S_2 in the proof of Theorem 1.2 of [Ju4], but in the end the two error terms of Theorem 33 dominate all the other error terms.

For shorter sums we have the following theorem, similar to Theorem 6 in [V].

Theorem 35. *Let $M \in [1, \infty[$, let $\Delta \in \mathbb{R}_+$ with $M^\varepsilon \ll \Delta \ll M^{1/2-\varepsilon}$, and let h and k be coprime integers with $1 \leqslant k \ll \Delta^{1/2-\varepsilon} M^{-\vartheta}$. Then*

$$\int\limits_M^{2M} \Bigg| \sum_{x \leqslant n \leqslant x+\Delta} t(n)\, e\!\left(\frac{nh}{k}\right) \Bigg|^2 \mathrm{d}x \asymp M\,\Delta.$$

In particular, the sum in question is $\Omega(\Delta^{1/2})$ when $\Delta \gg M^{2\vartheta+\varepsilon}$.

This follows for instance by following the proof of Theorem 6 in [V]: the main difference is that the error terms $k^2\, \Xi\, M^\varepsilon + k\, \Xi\, \Delta^{1/2} M^\varepsilon$ are to be replaced by $k^2\, \Xi\, M^{2\vartheta+\varepsilon} + k\, \Xi\, \Delta^{1/2} M^{\vartheta+\varepsilon}$. Except for the error term of the truncated Voronoi identity, the exponent ϑ never appears as the Fourier coefficients are estimated by the Rankin–Selberg estimate.

Acknowledgements

The authors would like to express their deep gratitude for the encouragement and support of Dr. A.-M. Ernvall-Hytönen. The authors would also like to gratefully acknowledge the many beneficial and thoughtful comments of an anonymous referee.

During this research, the first author was funded by the Academy of Finland project Number Theory Finland and the Doctoral Programme for Mathematics and Statistics of the University of Helsinki. The second author was funded by Finland's Ministry of Education through the Doctoral School of Inverse Problems, Academy of Finland's Centre of Excellence in Inverse Problems Research and the Foundation of Vilho, Yrjö and Kalle Väisälä.

References

[B] J. Bourgain, *Decoupling, exponential sums and the Riemann zeta function*, J. Amer. Math. Soc. 30 (2017), 205–224.

[C] H. Cramér, *Über zwei Sätze von Herrn G. H. Hardy*, Math. Z. 15 (1922), 200–210.

[D] P. Deligne, *La conjecture de Weil: I*, Publ. Math. Inst. Hautes Études Sci. 43 (1974), 273–307.

[EHS] C. Epstein, J. L. Hafner and P. Sarnak, *Zeros of L-functions attached to Maass forms*, Math. Z. 190 (1985), 113–128.

[E1] A.-M. Ernvall-Hytönen, *On the error term in the approximate functional equation for exponential sums related to cusp forms*, Int. J. Number Theory 4 (2008), 747–756.

[E2] A.-M. Ernvall-Hytönen, *A relation between Fourier coefficients of holomorphic cusp forms and exponential sums*, Publ. Inst. Math. (Beograd) (N.S.) 86 (2009), 97–105.

[E3] A.-M. Ernvall-Hytönen, *A note concerning certain exponential sums related to cusp forms*, Šiauliai Math. Semin. 4 (2009), 75–82.

[E4] A.-M. Ernvall-Hytönen, *On certain exponential sums related to* GL(3) *cusp forms*, C. R. Acad. Sci. Paris, Ser. I 348 (2010), 5–8.

[E5] A.-M. Ernvall-Hytönen, *On the mean square of short exponential sums related to cusp forms*, Funct. Approx. Comment. Math. 45 (2011), 97–104.

[E6] A.-M. Ernvall-Hytönen, *Mean square estimate for relatively short exponential sums involving Fourier coefficients of cusp forms*, Ann. Acad. Sci. Fenn. Math. 40 (2015), 385–395.

[E7] A.-M. Ernvall-Hytönen, personal communication.

[EK] A.-M. Ernvall-Hytönen and K. Karppinen, *On short exponential sums involving Fourier coefficients of holomorphic cusp forms*, Int. Math. Res. Not. IMRN article ID rnn022 (2008), 1–44.

[EJV] A.-M. Ernvall-Hytönen, J. Jääsaari and E. V. Vesalainen, *Resonances and* Ω*-results for exponential sums related to Maass forms for* $\mathrm{SL}(n, \mathbb{Z})$, J. Number Theory 153 (2015), 135–157.

[G] D. Godber, *Additive twists of Fourier coefficients of modular forms*, J. Number Theory 133 (2013), 83–104.

[GL1] D. Goldfeld and X. Li, *Voronoi formulas on* $\mathrm{GL}(n)$, Int. Math. Res. Not. IMRN article ID 86295 (2006), 1–25.

[GL2] D. Goldfeld and X. Li, *The Voronoi formula for* $\mathrm{GL}(n, \mathbb{R})$, Int. Math. Res. Not. IMRN article ID rnm144 (2008), 1–39.

[GH] K. Győry and G. Halász (eds.), *Number Theory Vol. I. Elementary and Analytic*, Colloquia Mathematica Societatis János Bolyai, 51, North-Holland Publishing Company, 1990.

[H] J. L. Hafner, *Some remarks on odd Maass wave forms (and a correction to [EHS])*, Math. Z. 196 (1987), 129–132.

[HI] J. L. Hafner and A. Ivić, *On sums of Fourier coefficients of cusp forms*, Enseign. Math. 35 (1989), 375–382.

[H1] M. N. Huxley, *Area, Lattice Points and Exponential Sums*, London Mathematical Society Monographs, New Series, 13, Oxford University Press, 1996.

[H2] M. N. Huxley, *Exponential sums and the Riemann zeta function V*, Proc. London Math. Soc 90 (2005), 1–41.

[Iv1] A. Ivić, *The Riemann Zeta-Function: Theory and Applications*, Dover Publications, 2003.

[Iv2] A. Ivić, *The circle and divisor problem*, Bull. Cl. Sci. Math. Nat. Sci. Math. 29 (2004), 79–83.

[Iv3] A. Ivić, *On the divisor function and the Riemann zeta-function in short intervals*, Ramanujan J. 19 (2009), 207–224.

[Iw] H. Iwaniec, *Spectral Methods of Automorphic Forms*, Graduate Studies in Mathematics, 53, American Mathematical Society, 2002.

[Ji] L. Ji (ed.), *Geometry and Analysis, No. 2*, Advanced Lectures in Mathematics, 18, International Press, 2011.

[Ju1] M. Jutila, *Riemann's zeta-function and the divisor problem*, Ark. Mat. 21 (1983), 75–96.

[Ju2] M. Jutila, *On the divisor problem for short intervals*, Ann. Univ. Turku. Series AI 186 (1984), 23–30.

[Ju3] M. Jutila, *On exponential sums involving the divisor function*, J. Reine Angew. Math. 355 (1985), 173–190.

[Ju4] M. Jutila, *Lectures on a Method in the Theory of Exponential Sums*, Lectures on Mathematics, 80, Tata Institute of Fundamental Research, 1987.

[Ju5] M. Jutila, *On exponential sums involving the Ramanujan function*, Proc. Indian Acad. Sci. 97 (1987), 157–166.

[Ju6] M. Jutila, *Mean value estimates for exponential sums*, in [SW], 120–136.

[Ju7] M. Jutila, *The fourth power moment of the Riemann zeta-function over a short interval*, in [GH], 221–244.

[JM] M. Jutila and Y. Motohashi, *Uniform bound for Hecke L-functions*, Acta Math. 195 (2005), 61–115.

[Ka] K. Karppinen, *Kärkimuotoihin liittyvät eksponenttisummat* (*Exponential sums related to cusp forms*, in Finnish), licentiate's thesis, University of Turku, 1998.

[Ki] H. H. Kim, *Functoriality for the exterior square of* GL_4 *and the symmetric fourth of* GL_2, J. Amer. Math. Soc. 16 (2003), 139–183.

[KS] H. H. Kim and P. Sarnak, *Appendix 2: Refined estimates towards the Ramanujan and Selberg conjectures*, J. Amer. Math. Soc. 16 (2003), 175–181.

[Le] N. N. Lebedev, *Special Functions & their Applications*, Dover Publications, 1972.

[LY] X. Li and M. P. Young, *Additive twists of Fourier coefficients of symmetric-square lifts*, J. Number Theory 132 (2012), 1626–1640.

[Lü] G. Lü, *On averages of Fourier coefficients of Maass cusp forms*, Arch. Math. (Basel) 100 (2013), 255–265.

[Me1] T. Meurman, *On the order of the Maass L-functions on the critical line*, in [GH], 325–354.

[Me2] T. Meurman, *On exponential sums involving the Fourier coefficients of Maass wave forms*, J. Reine Angew. Math. 384 (1988), 192–207.

[Mi] S. D. Miller, *Cancellation in additively twisted sums on* $\mathrm{GL}(n)$, Amer. J. Math. 128 (2006), 699–729.

[MS1] S. D. Miller and W. Schmid,*Automorphic distributions, L-functions, and Voronoi summation for* $\mathrm{GL}(3)$, Ann. Math. 164 (2006), 423–488.

[MS2] S. D. Miller and W. Schmid, *A general Voronoi summation formula for* $\mathrm{GL}(n, \mathbb{Z})$, in [Ji], 173–224.

[Mo] Y. Motohashi, *Spectral Theory of the Riemann Zeta-Function*, Cambridge Tracts in Mathematics, 127, Cambridge University Press, 1997.

[R] R. A. Rankin, *Contributions to the theory of Ramanujan's function* $\tau(n)$ *and similar arithmetical functions II. The order of Fourier coefficients of integral modular forms*, Math. Proc. Cambridge Phil. Soc. 35 (1939), 357–372.

[SW] H. P. Schlickewei and E. Wirsing (eds.), *Number Theory: Proceedings of the Journées Arithmetiques held in Ulm, FRG, September 14–18, 1987*, Lecture Notes in Mathematics, 1380, Springer, 1989.

[S] A. Selberg, *Bemerkungen über eine Dirichletsche Reihe, die mit der Theorie der Modulformen nahe verbunden ist*, Arch. Math. Naturvid. 43 (1940), 47–50.

[T] K.-M. Tsang, *Recent progress on the Dirichlet divisor problem and the mean square of the Riemann zeta-function*, Sci. China Math. 53 (2010), 2561–2572.

[V] E. V. Vesalainen, *Moments and oscillations of exponential sums related to cusp forms*, Math. Proc. Cambridge Philos. Soc. 162 (2017), 479–506.

[W1] J. R. Wilton, *A note on Ramanujan's function* $\tau(n)$, Math. Proc. Cambridge Philos. Soc. 25 (1929), 121–129.

[W2] J. R. Wilton, *An approximate functional equation with applications to a problem of Diophantine approximation*, J. Reine Angew. Math. 169 (1933), 219–237.