

# ON THE NUMBER OF HECKE EIGENVALUES OF SAME SIGN ON $GL_n$

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**ABSTRACT.** We study the distribution of signs of the (real-valued) Hecke eigenvalues  $A(m, 1, \dots, 1)$  of self-dual Hecke–Maass cusp forms for the group  $SL_n(\mathbb{Z})$ , where  $n \geq 2$  is an integer. Our main result establishes, under Generalised Ramanujan–Petersson Conjecture, that for almost all  $x \sim X$  the short interval  $[x, x+H]$  contains a subset  $\mathcal{S}$  (resp.  $\mathcal{S}'$ ) of size  $\gg H(\log X)^{1/n^2-1}$  such that  $A(m, 1, \dots, 1)$  is positive (resp. negative) for all  $m \in \mathcal{S}$  (resp.  $m \in \mathcal{S}'$ ), provided that  $(\log X)^{1-1/n^2} \ll H \ll X$ . We also prove a slightly stronger result unconditionally for  $GL_2$  and  $GL_3$  Hecke–Maass cusp forms. In addition, we obtain results under weaker bounds towards Generalised Ramanujan–Petersson Conjecture. Finally, as a by-product of our methods, we improve earlier bounds for the number of Hecke eigenvalues of same sign also in long intervals unconditionally for  $GL_2$  and  $GL_3$  Hecke–Maass cusp forms, and under Generalised Ramanujan–Petersson Conjecture for  $GL_n$  forms when  $n \geq 4$ .

## 1. INTRODUCTION

The distribution of Hecke eigenvalues is a central question in the theory of automorphic forms. During the past few decades studying statistical properties (such as sizes and sign changes) of Hecke eigenvalues of automorphic forms has attracted much attention, see e.g. [14, 15, 35, 18, 16, 24, 25]. The interest in signs of these eigenvalues partly stems from their connections to other questions such as those concerning the distribution of zeroes of automorphic forms [3, 19, 8, 9] and partly them being higher rank analogues of real characters. Let  $f$  be a Maass cusp form for the group  $SL_n(\mathbb{Z})$  with Fourier coefficients  $A(m_1, \dots, m_{n-1})$ . Assume further that the form is an eigenfunction of the full Hecke algebra and normalised so that  $A(1, \dots, 1) = 1$ . In this case the Hecke eigenvalue under the  $m^{\text{th}}$  Hecke operator is given by  $A(m, 1, \dots, 1)$ .

In this paper we focus on the signs of Hecke eigenvalues. When  $n > 2$  the Hecke eigenvalues  $A(m, 1, \dots, 1)$  are not necessarily real-valued, but this property is satisfied when the underlying form is self-dual. We impose this assumption throughout the paper. For the majority of this work we use the notation  $A(m) := A(m, 1, \dots, 1)$  for the Hecke eigenvalues<sup>1</sup>. Rather than studying sign changes, the aim of this work is to study the question that how many Hecke eigenvalues of the same sign a given interval contains?

To study this we define the counting function

$$\mathcal{N}_f^\pm(x) := \sum_{\substack{m \leq x \\ A(m) \gtrless 0}} 1,$$

where “+” corresponds to  $A(m) > 0$  and “−” corresponds to  $A(m) < 0$ .

When  $n = 2$ , Lau and Wu [17] have shown that the optimal lower bound  $\mathcal{N}_f^\pm(x) \gg x$  holds in the case where  $f$  is a holomorphic cusp form. For Maass cusp forms (with any rank  $n \geq 2$ ) a weaker lower bound  $\mathcal{N}_f^\pm(x) \gg x^{1-\vartheta_n}(\log x)^{2/n-2}$  has been obtained by Liu and Wu [20] assuming<sup>2</sup> Hypothesis H of Rudnick and Sarnak [29]. Here  $\vartheta_n$  is the exponent towards Generalised Ramanujan–Petersson Conjecture for Hecke–Maass cusp forms on  $GL_n$ . The main reason for this weaker result in the setting of (higher rank) Maass cusp forms is that the method of  $\mathcal{B}$ -free numbers used in the case of holomorphic cusp forms is difficult to generalise as it uses Serre’s estimate [31] on the size of the set  $\{p \leq x : A(p) = 0\}$  as an input, a analogue of this result

2020 *Mathematics Subject Classification*. Primary 11F30; Secondary 11F12.

<sup>1</sup>Sometimes we emphasise the underlying form  $f$  by writing  $A_f(m)$ .

<sup>2</sup>In fact their result is unconditional for  $2 \leq n \leq 4$ .

for (higher rank) Maass cusp forms is not known and is a challenging open problem. Indeed, assuming the non-lacunarity property

$$\sum_{\substack{p \\ A(p)=0}} \frac{1}{p} < \infty$$

(which follows from a generalisation of Serre's result), the  $\mathcal{B}$ -free number argument of Lau and Wu [17] would give the expected lower bound  $\mathcal{N}_f^\pm(x) \gg x$  for arbitrary rank  $n$  Hecke–Maass cusp forms. Nevertheless, Liu and Wu [20] were able to obtain a non-trivial lower bound by different means.

It is natural to wonder whether it is also possible to show that shorter intervals contain plenty of both positive and negative Hecke eigenvalues. In the classical situation ( $n = 2$ ) Lau and Wu were able to obtain such a result for intervals of length constant times  $x^{1/2}$ . Indeed, they showed [17, Theorem 2] that there exists a constant  $C$  depending on  $f$  so that for sufficiently large  $x$  and any  $\varepsilon > 0$  we have<sup>3</sup>

$$\mathcal{N}_f^\pm(x + Cx^{1/2}) - \mathcal{N}_f^\pm(x) \gg_\varepsilon x^{1/4-\varepsilon}.$$

As far as the author is aware of, no prior results of this type are known in the higher rank setting. The main goal of the present work is to provide such a result for general Hecke–Maass cusp forms under Generalised Ramanujan–Petersson Conjecture. Let us set

$$\Delta_f(X) := \prod_{\substack{p \leq X \\ A_f(p)=0}} \left(1 - \frac{1}{p}\right).$$

Observe that (see the proof of Theorem 1.2) under Generalised Ramanujan–Petersson Conjecture we have

$$\#\{m \sim X : A_f(m) \neq 0\} \asymp_f X \Delta_f(X).$$

**Remark 1.1.** Also note that  $\Delta_f(X)$  is the density parameter  $\delta(\mathcal{N}; X)$  appearing in [26] for the multiplicative set<sup>4</sup>  $\mathcal{N} = \{m \in \mathbb{N} : A_f(m) \neq 0\}$ .

Our first result gives precise information about the number of positive and negative Hecke eigenvalues for Hecke–Maass cusp forms for the group  $\mathrm{GL}_n(\mathbb{R})$  in short intervals  $[x, x + H]$  for almost all  $x$  in a wide range of  $H$ .

**Theorem 1.2.** *Let  $f$  be a Hecke–Maass cusp form for the group  $\mathrm{SL}_n(\mathbb{Z})$ . Assume Generalised Ramanujan–Petersson Conjecture and let  $1 \leq H \leq X/10$  be so that  $H \Delta_f(X) \rightarrow \infty$  as  $X \rightarrow \infty$ . Then for all but  $o_f(X)$  of  $x \sim X$  we have that*

$$\mathcal{N}_f^\pm(x + H) - \mathcal{N}_f^\pm(x) = \frac{H}{2X} \#\{m \sim X : A_f(m) \neq 0\} + o_f(H \Delta_f(X)).$$

In particular,

$$\mathcal{N}_f^\pm(x + H) - \mathcal{N}_f^\pm(x) \asymp_f H \Delta_f(X)$$

for almost all  $x \sim X$ .

It is also easy to show (see the end of Section 5) that under Generalised Ramanujan–Petersson Conjecture we have  $\Delta_f(X) \gg_f (\log X)^{-1+1/n^2}$  and so we have the estimate

$$\mathcal{N}_f^\pm(x + H) - \mathcal{N}_f^\pm(x) \gg_f \frac{H}{(\log X)^{1-1/n^2}}$$

for almost all  $x \sim X$  in the same range of  $H$ .

We are also able to prove slightly sharper results for  $\mathrm{GL}_2$  and  $\mathrm{GL}_3$  Hecke–Maass cusp forms without assuming Generalised Ramanujan–Petersson Conjecture.

<sup>3</sup>Throughout the paper, when we write  $\mathcal{N}_f^\pm(\dots) - \mathcal{N}_f^\pm(\dots)$  we assume that the choices  $\{+, -\}$  in upper indices coincide.

<sup>4</sup>More generally, we say that a subset  $\mathcal{N} \subset \mathbb{N}$  is *multiplicative* if for any  $m, n \geq 1$  with  $(m, n) = 1$  we have  $m, n \in \mathcal{N}$  if and only if  $mn \in \mathcal{N}$ .

**Theorem 1.3.** (1) Let  $f$  be a Hecke–Maass cusp form for the group  $\mathrm{SL}_2(\mathbb{Z})$ . Then for  $\sqrt{\log X} \ll H \leq X/10$  we have

$$\mathcal{N}_f^\pm(x+H) - \mathcal{N}_f^\pm(x) \gg_f \frac{H}{\sqrt{\log X}}$$

for almost all  $x \sim X$ .

(2) Let  $f$  be a Hecke–Maass cusp form for the group  $\mathrm{SL}_3(\mathbb{Z})$ . Then for  $(\log X)^{2/3} \ll H \leq X/10$  we have

$$\mathcal{N}_f^\pm(x+H) - \mathcal{N}_f^\pm(x) \gg_f \frac{H}{(\log X)^{2/3}}$$

for almost all  $x \sim X$ .

Note that in the  $n = 2$  case this is much stronger than the result<sup>5</sup> of Lau and Wu mentioned above, of course with the caveat that our result holds for almost all  $x$ .

**Remark 1.4.** By inspecting the proof one observes that if the asymptotics

$$\sum_{p \leq x} A(p)^4 \log p \sim C_4 x$$

for the fourth moment is known for some absolute constant  $C_4 > 0$ , then the conclusions of the previous theorems hold in the range  $H \gg (\log X)^{1-1/C_4+o(1)}$ . For  $\mathrm{GL}_3$  Hecke–Maass cusp forms the self-duality happens to make  $C_4 = 3$  accessible without assuming Generalised Ramanujan–Petersson Conjecture using currently known results towards Langlands functoriality.

As a direct consequence of Theorem 1.2 we deduce the following.

**Corollary 1.5.** Let  $f$  be a Hecke–Maass cusp form for the group  $\mathrm{SL}_n(\mathbb{Z})$ . Assume Generalised Ramanujan–Petersson Conjecture. Then, provided that  $(\log X)^{1-1/n^2} \ll H \leq X/10$ , there are  $\gg X/H$  pairwise disjoint subintervals on  $[X, 2X]$  such that if  $x$  belongs to any of these subintervals, we have

$$(1.1) \quad \mathcal{N}_f^\pm(x+H) - \mathcal{N}_f^\pm(x) \gg_f \frac{H}{(\log X)^{1-1/n^2}}.$$

Furthermore, by Theorem 1.3 the assumption of Generalised Ramanujan–Petersson Conjecture may be removed and the lower bound in (1.1) may be slightly increased for  $n \in \{2, 3\}$ . These deductions are made in Section 7. In the case  $n = 2$  this significantly improves an earlier result<sup>6</sup> of Wu and Zhai [37, Theorems 3 and 4] achieved by a different method. Furthermore, as far as the author is aware of, no prior results of this kind existed in the higher rank setting.

As a by-product of these arguments we are able to produce lower bounds for the quantity  $\mathcal{N}_f^\pm(x)$  for any  $x$ .

**Theorem 1.6.** Let  $f$  be a Hecke–Maass cusp form for the group  $\mathrm{SL}_n(\mathbb{Z})$ . Assume Generalised Ramanujan–Petersson Conjecture. Then we have

$$\mathcal{N}_f^\pm(x) \gg \frac{x}{(\log x)^{1-1/n^2}}$$

for sufficiently large  $x$ .

The result of Liu and Wu [20] implies that  $\mathcal{N}_f^\pm(x) \gg x(\log x)^{2/n-2}$  under Generalised Ramanujan–Petersson Conjecture and so Theorem 1.6 improves their result by a logarithmic factor.

We can also improve the lower bound for the quantity  $\mathcal{N}_f^\pm(x)$  without assuming Generalised Ramanujan–Petersson Conjecture for  $\mathrm{GL}_2$  and  $\mathrm{GL}_3$  forms.

<sup>5</sup>Their result was formulated for holomorphic cusp forms, but under Generalised Ramanujan–Petersson Conjecture their method gives an analogous result for  $\mathrm{GL}_2$  Hecke–Maass cusp forms.

<sup>6</sup>Again, technically their result is for holomorphic cusp forms, but the argument can be adapted to the Maass cusp form setting as well.

**Theorem 1.7.** (1) Let  $f$  be a Hecke–Maass cusp form for the group  $\mathrm{SL}_2(\mathbb{Z})$ . Then we have

$$\mathcal{N}_f^\pm(x) \gg_f \frac{x}{\sqrt{\log x}}$$

for sufficiently large  $x$ .

(2) Let  $f$  be a Hecke–Maass cusp form for the group  $\mathrm{SL}_3(\mathbb{Z})$ . Then we have

$$\mathcal{N}_f^\pm(x) \gg_f \frac{x}{(\log x)^{2/3}}$$

for sufficiently large  $x$ .

The first part slightly improves the bound  $\mathcal{N}_f^\pm(x) \gg_f x/\log x$ , which follows from [25, Theorem 1.2]. The second part significantly improves an earlier results of Liu and Wu [20] who showed  $\mathcal{N}_f^\pm(x) \gg_f x^{2/7}(\log x)^{-4/3}$  for  $\mathrm{GL}_3$  Maass cusp forms. Observe also that for a holomorphic cusp form  $f$ , Lau, Liu, and Wu [18] have shown that  $\mathcal{N}_{\mathrm{Sym}^2 f}^\pm(x) \gg_f x$ . However, the method that achieves this is based on  $\mathcal{B}$ -free numbers and thus relies on Deligne’s bound for the Hecke eigenvalues that is not presently available for Maass cusp forms.

**Remark 1.8.** (1) If one considers Maass cusp forms of higher level, then in the  $\mathrm{GL}_2$  case such a form may correspond to a representation of dihedral type. For these forms one can show a slightly weaker bound

$$\mathcal{N}_f^\pm(x) \gg_f \frac{x}{(\log x)^{3/4}}.$$

(2) In the  $\mathrm{GL}_3$  case one can in turn prove a slightly improved bound

$$\mathcal{N}_f^\pm(x) \gg_f \frac{x}{(\log x)^{1/3}}.$$

if the form  $f$  is a Gelbart–Jacquet lift of a  $\mathrm{GL}_2$  Maass cusp form that is of octahedral type.

We also apply another method based on the moments of sums of Hecke eigenvalues to obtain results that hold under a weak Generalised Ramanujan–Petersson Conjecture, but on the other hand simultaneously assuming Generalised Lindelöf Hypothesis for the standard Godement–Jacquet  $L$ -function and the Rankin–Selberg  $L$ -function.

**Theorem 1.9.** Let  $f$  be a Hecke–Maass cusp form for the group  $\mathrm{SL}_n(\mathbb{Z})$ . Assume the weak Ramanujan–Petersson Conjecture in the form  $|A(m)| \ll_\varepsilon m^{\vartheta+\varepsilon}$ . Furthermore, assume Generalised Lindelöf Hypothesis for  $L(s, f)$  and the Rankin–Selberg  $L$ -function  $L(s, f \times f)$ .

(1) Let  $1 \leq H \ll X$ . Then there exists a subset  $\mathcal{E}(X, H)$  with  $|\mathcal{E}(X, H)| \ll_\varepsilon X^{1+2\vartheta+\varepsilon}/H$  so that for any  $x \in [X, 2X] \setminus \mathcal{E}(X, H)$  we have

$$\mathcal{N}_f^\pm(x+H) - \mathcal{N}_f^\pm(x) \gg_\varepsilon H X^{-2\vartheta-\varepsilon}.$$

In particular, for  $H \geq X^{2\vartheta+\varepsilon}$  the exceptional set has size  $o(X)$ .

(2) For  $H \geq X^{1/2+\vartheta+\eta}$ , with an arbitrarily small fixed  $\eta > 0$ , we have

$$\mathcal{N}_f^\pm(x+H) - \mathcal{N}_f^\pm(x) \gg_\varepsilon H X^{-2\vartheta-\varepsilon}$$

for any  $x \sim X$ .

Finally, assuming the full Ramanujan–Petersson Conjecture we are able to sharpen the factor  $X^{-\varepsilon}$  to a logarithmic factor in part (2) of the previous theorem.

**Theorem 1.10.** Let  $f$  be a Hecke–Maass cusp form for the group  $\mathrm{SL}_n(\mathbb{Z})$ . Suppose that  $x^{1/2+\varepsilon} \leq H \ll x$ . Assume Generalised Ramanujan–Petersson Conjecture and Generalised Lindelöf Hypothesis for both  $L(s, f)$  and  $L(s, f \times f)$ . Then we have

$$\mathcal{N}_f^\pm(x+H) - \mathcal{N}_f^\pm(x) \gg_\varepsilon \frac{H}{(\log x)^{2 \log n + \varepsilon}}.$$

## 2. THE STRATEGY

We make a few remarks concerning the proofs. For Theorem 1.2 our methods largely build upon the work of Matomäki and Radziwiłł [26] concerning multiplicative functions in short intervals. We consider the multiplicative functions  $g(m) = \text{sgn}(A(m))$  and  $r(m) = |g(m)|$ , and apply [26, Corollary 1.1] to reduce the proof to understanding averages of  $g(m)$  and  $r(m)$  over dyadic intervals. At this stage we are able to use some arguments inspired by the work of Mangerel [23]. Much of the work involves verifying conditions needed to apply certain results from the theory of multiplicative functions. This turns out not to be entirely straightforward.

To remove Generalised Ramanujan–Petersson Conjecture when  $n \in \{2, 3\}$  (Theorem 1.3) the key observation is that such an assumption can be replaced with an estimate of the form, for some absolute constant  $C_4 > 0$ ,

$$\sum_{z < p \leq w} \frac{|A(p)|^4}{p} \leq C_4 \sum_{z < p \leq w} \frac{1}{p} + O_f\left(\frac{1}{\log z}\right)$$

uniformly in  $z < p \leq w$ . As Mangerel notes [22, Remark 1.3], such an estimate is not generally available unconditionally for  $n \geq 3$ . However, we show that using known results from the theory of automorphic forms it is possible to obtain such a result without Generalised Ramanujan–Petersson Conjecture for self-dual  $f$  in the case  $n = 3$ . This relies particularly on the description of self-dual Maass cusp forms for the group  $SL_3(\mathbb{Z})$  due to Ramakrishnan [28].

The proof of Theorem 1.6 is a simple application of Wirsing’s theorem and multiplicativity of  $A(m)$  in the form that if  $r$  is such that  $A(r) < 0$ , then  $A(m)$  and  $A(mr)$  have opposite signs for  $(m, r) = 1$ . Again removing the assumption of Generalised Ramanujan–Petersson Conjecture for  $n \in \{2, 3\}$  (Theorem 1.7) is based on the observation that such an input can be replaced with

$$\sum_{\substack{z < p \leq w \\ A(p) \neq 0}} \frac{1}{p} \geq \alpha \sum_{z < p \leq w} \frac{1}{p} - O\left(\frac{1}{\log z}\right),$$

which can then be verified for  $n \in \{2, 3\}$ . Such an assumption allows us again to apply results from [26].

Theorem 1.9 relies on estimates for

$$\sum_{x \leq m \leq x+H} A(m) \quad \text{and} \quad \sum_{x \leq m \leq x+H} |A(m)|^2.$$

These are needed both for individual  $x$  and for averages over  $x \sim X$ . Such bounds are achieved by nowadays standard contour integration techniques. We assume Generalised Lindelöf Hypothesis for simplicity, but a weaker pointwise bound for  $L(1/2 + it, f)$  should also suffice. Finally, the results of the previous theorem can be somewhat sharpened (Theorem 1.10) if one assumes Generalised Ramanujan–Petersson Conjecture. This is a fairly straightforward consequence of Shiu’s bound and the observation that Generalised Ramanujan–Petersson Conjecture implies the estimate  $|A(p)|^q \leq n^{q-2}|A(p)|^2$  for any  $q > 2$ .

**2.1. Acknowledgements.** This work was supported by the Finnish Cultural Foundation and the Emil Aaltonen Foundation. The author wishes to thank Kaisa Matomäki for interesting discussions concerning sign changes and multiplicative functions throughout the years, and in particular for pointing out the reference [26]. He is also grateful to Steve Lester for useful suggestions.

**2.2. Declaration on the use of AI.** ChatGPT 5.6 was used to identify relevant literature (which in particular led to locating the reference [13]) and to proofread the manuscript. The output from proofreading gave rise to Remark 1.8 and Theorem 1.10, which were carefully reviewed, verified, and revised by the author, who takes full responsibility for their content. Otherwise rest of the text and mathematical arguments in this paper are entirely human generated.

## 3. ORGANISATION OF THE PAPER

This paper is organised as follows. In Section 5 we gather the tools needed in the proofs. Theorem 1.2 is proved in Section 6. Section 7 is devoted to the proof of Theorem 1.3 and the deduction of Corollary 1.5. Section 8 occupies the proofs of Theorems 1.6 and Theorem 1.7. The final section completes the proofs of Theorems 1.9 and 1.10.

## 4. NOTATION

We use standard asymptotic notation. If  $f$  and  $g$  are complex-valued functions defined on some set, say  $\mathcal{G}$ , then we write  $f \ll g$  to signify that  $|f(x)| \leq C|g(x)|$  for all  $x \in \mathcal{G}$  for some implicit constant  $C \in \mathbb{R}_+$ . The notation  $O(g)$  denotes a quantity that is  $\ll g$ , and  $f \asymp g$  means that both  $f \ll g$  and  $g \ll f$ . We write  $f = o(g)$  if  $g$  never vanishes in  $\mathcal{G}$  and  $f(x)/g(x) \rightarrow 0$  as  $x \rightarrow \infty$ . Moreover, we write<sup>7</sup>  $f \sim g$  if  $f(x)/g(x) \rightarrow 1$  as  $x \rightarrow \infty$ . The letter  $\varepsilon$  denotes a positive real number, whose value can be fixed to be arbitrarily small, and whose value can be different in different instances in a proof. All implicit constants are allowed to depend on  $\varepsilon$ , on the implicit constants appearing in the assumptions of theorem statements, and on anything that has been fixed. When necessary, we will use subscripts  $\ll_{\alpha, \beta, \dots}, O_{\alpha, \beta, \dots}$ , etc. to indicate when implicit constants are allowed to depend on quantities  $\alpha, \beta, \dots$ .

Let us also write  $1_{\mathcal{A}}(x)$  for the characteristic function for  $x \in \mathcal{A}$ . Furthermore,  $\operatorname{Re}(s)$  and  $\operatorname{Im}(s)$  are the real- and imaginary parts of  $s \in \mathbb{C}$ , respectively, and occasionally we write  $\sigma$  for  $\operatorname{Re}(s)$ . We write  $e(x) := e^{2\pi i x}$ . As usual,  $\mu$  denotes the Möbius function,  $\mathbb{A}_{\mathbb{Q}}$  denotes the adèle ring of  $\mathbb{Q}$ , and  $\operatorname{sgn}(m)$  denotes the sign of an integer  $m$ . Also  $\zeta$  is the Riemann zeta function,  $\Gamma$  denotes the Gamma function, and  $\gamma$  is the Euler–Mascheroni constant.

## 5. PRELIMINARIES

**5.1. Automorphic forms.** Let  $f$  be a Hecke–Maass cusp form of type  $(\nu_1, \dots, \nu_{n-1}) \in \mathbb{C}^{n-1}$  for the group  $\operatorname{SL}_n(\mathbb{Z})$  with Fourier coefficients  $A(m_1, \dots, m_{n-1})$ . If the form is normalised so that  $A(1, \dots, 1) = 1$ , then the eigenvalue under the  $m^{\text{th}}$  Hecke operator is given by  $A(m, 1, \dots, 1)$ . The key property is that the Hecke eigenvalues  $A(m, 1, \dots, 1)$  are multiplicative [4, Theorem 9.3.11].

Next we define the notion of a dual Maass cusp form. Let

$$\tilde{f}(z) := f(w \cdot {}^t(z^{-1})w), \quad \text{where} \quad w := \begin{pmatrix} & & & (-1)^{\lfloor \frac{n}{2} \rfloor} \\ & & 1 & \\ & \ddots & & \\ 1 & & & \end{pmatrix}.$$

Then  $\tilde{f}$  is a Maass cusp form of type  $(\nu_{n-1}, \dots, \nu_1) \in \mathbb{C}^{n-1}$  for the group  $\operatorname{SL}_n(\mathbb{Z})$  and it is called the dual Hecke–Maass cusp form of  $f$ . It turns out that

$$A_f(m_1, \dots, m_{n-1}) = A_{\tilde{f}}(m_{n-1}, \dots, m_1)$$

for every  $m_1, \dots, m_{n-1} \geq 1$ . For Hecke eigenforms a simple application of Möbius inversion shows that  $A(m_1, \dots, m_{n-1}) = \overline{A(m_{n-1}, \dots, m_1)}$ . By combining the previous two observations we obtain

$$\overline{A_f(m, 1, \dots, 1)} = A_{\tilde{f}}(m, 1, \dots, 1).$$

In particular, if the form  $f$  is self-dual ( $f = \tilde{f}$ ), then the Hecke eigenvalues are real-valued.

Associated to such a form  $f$  is the  $L$ -series given by

$$L(s, f) := \sum_{m=1}^{\infty} \frac{A(m, 1, \dots, 1)}{m^s},$$

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<sup>7</sup>This should not be confused with the notation  $\ell \sim L$  used also for  $L \leq \ell \leq 2L$  in this paper.

which converges for  $\mathrm{Re}(s) > 1$ . This has an entire continuation to the whole complex plane via the functional equation

$$L(s, f) = \pi^{ns-n/2} \frac{G(1-s, \tilde{f})}{G(s, f)} L(1-s, \tilde{f}),$$

where

$$G(s, f) := \prod_{j=1}^n \Gamma\left(\frac{s - \lambda_j(\nu)}{2}\right) \quad \text{and so} \quad G(s, \tilde{f}) := \prod_{j=1}^n \Gamma\left(\frac{s - \tilde{\lambda}_j(\nu)}{2}\right).$$

Here  $\lambda_j(\nu)$  and  $\tilde{\lambda}_j(\nu)$  are the Langlands parameters of  $f$  and  $\tilde{f}$ , respectively.

Generalised Lindelöf Hypothesis in the  $t$ -aspect states that on the critical line  $\sigma = 1/2$  one has the estimate  $L(1/2 + it, f) \ll_\varepsilon (1 + |t|)^\varepsilon$  for any  $\varepsilon > 0$ . The Rankin–Selberg  $L$ -function of two Hecke–Maass cusp forms  $f$  and  $g$  for the group  $\mathrm{SL}_n(\mathbb{Z})$  is given by

$$L(s, f \times g) := \zeta(ns) \sum_{m_1, \dots, m_{n-1} \geq 1} \frac{A_f(m_1, \dots, m_{n-1}) \overline{A_g(m_1, \dots, m_{n-1})}}{(m_1^{n-1} m_2^{n-2} \dots m_{n-1})^s},$$

which converges for  $\mathrm{Re}(s) > 1$ . This  $L$ -series has an analytic continuation to the whole complex plane if  $g \neq \tilde{f}$  and has a meromorphic continuation to  $\mathbb{C}$  with a simple pole at  $s = 1$  if  $g = \tilde{f}$ . If we set

$$\Lambda(s, f \times g) := \prod_{i=1}^n \prod_{j=1}^n \pi^{(-2 + \lambda_i(\nu_f) + \overline{\lambda_j(\nu_g)})/2} \Gamma\left(\frac{s - \lambda_i(\nu_f) - \overline{\lambda_j(\nu_g)}}{2}\right) L(s, f \times g),$$

then the functional equation

$$\Lambda(s, f \times g) = \Lambda(1-s, \tilde{f} \times \tilde{g})$$

holds; see [4, Theorem 12.1.4].

If  $L(s, f)$  has an Euler product representation

$$L(s, f) = \sum_{m=1}^{\infty} \frac{A(m, 1, \dots, 1)}{m^s} = \prod_p \prod_{j=1}^n (1 - \alpha_{j,p}(f) p^{-s})^{-1}$$

for large enough  $\mathrm{Re}(s)$ , and similar representation holds for  $g$  with parameters  $\alpha_{j,p}(g)$ , then also the Rankin–Selberg  $L$ -function can be written as an Euler product

$$L(s, f \times g) = \prod_p \prod_{k=1}^n \prod_{\ell=1}^n \left(1 - \alpha_{k,p}(f) \overline{\alpha_{\ell,p}(g)} p^{-s}\right)^{-1}.$$

Recall that here the complex numbers  $\alpha_{j,p}(f)$  are called the Satake parameters of the underlying Hecke–Maass cusp form  $f$ . Analytic properties of  $L(s, f \times \tilde{f})$  imply that

$$\sum_{m_1^{n-1} m_2^{n-2} \dots m_{n-1} \leq x} |A(m_1, m_2, \dots, m_{n-1})|^2 \sim r_f \cdot x$$

for some constant  $r_f > 0$  depending on  $f$  [4, Proposition 12.1.6, Remark 12.1.8]. As noted in [6, Theorem 6], we also have the estimate

$$(5.1) \quad \sum_{m \leq x} |A(m, 1, \dots, 1)|^2 \sim r_f \cdot H_f(1) \cdot x.$$

It was also noted in the proof of [6, Theorem 6] that for

$$D_f(s) := \sum_{m=1}^{\infty} \frac{|A(m, 1, \dots, 1)|^2}{m^s}$$

we have the factorisation

$$D_f(s) = L(s, f \times \tilde{f}) H_f(s)$$

where  $H_f(s)$  is holomorphic and bounded in the half-plane  $\operatorname{Re}(s) > 1/2 + \vartheta$ . Here  $\vartheta$  is an exponent towards Generalised Ramanujan–Petersson Conjecture explained in the next paragraph.

This result can be interpreted as saying that the Fourier coefficients  $A(m_1, \dots, m_{n-1})$  are essentially of constant order of magnitude on average. However, known pointwise bounds for the Fourier coefficients are quite far from the expected truth. Generalised Ramanujan–Petersson Conjecture predicts that an estimate of the form  $A(m, 1, \dots, 1) \ll_\varepsilon m^\varepsilon$  holds for every  $\varepsilon > 0$ . There are however approximations towards this conjecture. Let  $\vartheta = \vartheta(n) \geq 0$  be the smallest non-negative real number so that the estimate  $A(m, 1, \dots, 1) \ll_\varepsilon m^{\vartheta+\varepsilon}$  holds. It is easy to see that  $\vartheta \leq 1/2$  [4, Proposition 12.1.6], but currently it is known that  $\vartheta \leq 1/2 - 1/(n^2 + 1)$ . This result is due to [10]. For small values of  $n$  sharper results are known. We have that  $\vartheta(2) \leq 7/64$ ,  $\vartheta(3) \leq 5/14$ , and  $\vartheta(4) \leq 9/22$  [21]. Generalised Ramanujan–Petersson Conjecture predicts that the value  $\vartheta(n) = 0$  is admissible for every  $n \geq 2$ . An equivalent estimate holds for the Satake parameters of the underlying form  $f$ . Namely, we have  $\alpha_{j,p}(f) \ll_\varepsilon p^{\vartheta(n)+\varepsilon}$  for every prime  $p$ .

Assuming Generalised Ramanujan–Petersson Conjecture, we have the automorphic prime number theorem, which says that there exists a positive constant  $c$  so that

$$\sum_{p \leq x} |A(p)|^2 \log p = x + O\left(xe^{-c\sqrt{\log x}}\right).$$

By partial summation this gives the uniform estimate

$$(5.2) \quad \sum_{z < p \leq w} \frac{|A(p)|^2}{p} = \sum_{z < p \leq w} \frac{1}{p} + O\left(\frac{1}{\log z}\right)$$

for  $2 \leq z < w$ . A detailed proof can be found in [22, Lemma 5.6]. We also have the estimate

$$(5.3) \quad \sum_{z < p \leq w} \frac{A(p)}{p} = O_f\left(\frac{1}{\log z}\right).$$

For certain lower rank forms the automorphic prime number theorem holds unconditionally. We record the following result of Wu and Ye [36, Theorem 3].

**Lemma 5.1.** *Let  $m, m'$  be positive integers. Let  $\pi$  be an automorphic representation of  $\operatorname{GL}_m(\mathbb{A}_{\mathbb{Q}})$  and  $\pi'$  be an automorphic representation of  $\operatorname{GL}_{m'}(\mathbb{A}_{\mathbb{Q}})$  with at least one of them being self-dual.*

(1) *Suppose that  $\max\{m, m'\} \leq 4$ . Then we have that*

$$(5.4) \quad \sum_{p \leq x} A_\pi(p) \overline{A_{\pi'}(p)} \log p = \begin{cases} \frac{x^{1+i\tau_0}}{1+i\tau_0} + O\left(xe^{-c\sqrt{\log x}}\right) & \text{if } \pi' \simeq \pi \otimes |\det|^{i\tau_0} \text{ for some } \tau_0 \in \mathbb{R} \\ O\left(xe^{-c\sqrt{\log x}}\right) & \text{otherwise} \end{cases}$$

*for some constant  $c > 0$ .*

(2) *If  $\max\{m, m'\} \geq 5$ , the asymptotic relation (5.4) holds under the Hypothesis  $H$  (for both  $\pi$  and  $\pi'$ ), saying that*

$$\sum_p \frac{(\log p)^2 |A_\pi(p^\nu)|^2}{p^\nu} < \infty$$

*for any fixed  $\nu \geq 2$ , with error term replaced by  $O(x/\log x)$ .*

It is well-known that under the Langlands correspondence cuspidal automorphic representations of  $\operatorname{GL}_2(\mathbb{A}_{\mathbb{Q}})$  correspond to two-dimensional continuous Galois representations  $\rho : \operatorname{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \rightarrow \operatorname{GL}_2(\mathbb{C})$ . The classification of these representations relies on the image of corresponding projective representations  $\bar{\rho} : \operatorname{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \rightarrow \operatorname{PGL}_2(\mathbb{C})$ . Thus automorphic representations of  $\operatorname{GL}_2(\mathbb{A}_{\mathbb{Q}})$  fall into distinct types corresponding to finite subgroups of  $\operatorname{PGL}_2(\mathbb{C})$ . These types are called *cyclic* (projective image is a cyclic group  $C_n$ ), *dihedral* (projective image is a dihedral group  $D_{2n}$ ), *tetrahedral* (projective image is the alternative group  $A_4$ ), *octahedral* (projective image is the symmetric group  $S_4$ ), and *icosahedral* (projective image is the alternative group  $A_5$ ).

**5.2. Moment bounds.** We begin by deriving certain second moment bounds under Generalised Lindelöf Hypothesis. For a closely related result (whose strategy we follow in the proof) for the  $k$ -fold divisor function, see [1, Theorem 1.3].

**Lemma 5.2.** *Let  $f$  be a Hecke–Maass cusp form for the group  $\mathrm{SL}_n(\mathbb{Z})$  and suppose that  $1 \leq H \ll X$ .*

- (1) *Assume Generalised Lindelöf Hypothesis for  $L(s, f)$  and  $L(s, f \times f)$  in the  $t$ -aspect. Let  $r_f$  be as in (5.1). Then we have*

$$\int_X^{2X} \left| \sum_{x \leq m \leq x+H} |a(m)|^2 - r_f H_f(1) \cdot H \right|^2 dx \ll_\varepsilon H X^{1+2\vartheta+\varepsilon}.$$

- (2) *Assume Generalised Lindelöf Hypothesis for  $L(s, f)$  in the  $t$ -aspect. Then we have*

$$\int_X^{2X} \left| \sum_{x \leq m \leq x+H} a(m) \right|^2 dx \ll_\varepsilon H X^{1+\varepsilon}.$$

*Proof.* The underlying mechanism in both of the estimates is similar. More generally, let  $c(m)$  be complex numbers so that

$$L(s) := \sum_{m=1}^{\infty} \frac{c(m)}{m^s}.$$

has a meromorphic continuation to the half-plane  $\mathrm{Re}(s) > \sigma_0$  (where  $0 < \sigma_0 < 1$ ) with a possible simple pole of residue  $c_f$  at  $s = 1$ . Furthermore, suppose that  $L(s)$  satisfies Generalised Lindelöf Hypothesis  $L(1/2 + it) \ll_\varepsilon (1 + |t|)^\varepsilon$  for every  $\varepsilon > 0$ . Using an inequality of Saffari and Vaughan [30, 5] we have

$$(5.5) \quad \int_X^{2X} \left| \sum_{x \leq m \leq x+H} c(m) - c_f H \right|^2 dx \ll \frac{X}{H} \int_0^{8H/X} \int_0^X \left| \sum_{x \leq m \leq (1+\beta)x} c(m) - c_f \beta x \right|^2 dx d\beta.$$

Now using Perron’s formula and shifting contour to the line  $\mathrm{Re}(s) = \sigma_0 + \varepsilon$  we have

$$\sum_{x \leq m \leq (1+\beta)x} c(m) - c_f \beta x = \frac{1}{2\pi i} \int_{(\sigma_0+\varepsilon)} L(s) x^s \frac{(1+\beta)^s - 1}{s} ds.$$

Next we observe that

$$\frac{(1+\beta)^{\sigma+it} - 1}{\sigma + it} \ll \min \left\{ \beta, \frac{1}{1+|t|} \right\}.$$

Thus smoothly decomposing the interval  $[X, 2X]$  and using Plancherel’s theorem we have

$$\int_X^{2X} \left| \sum_{x \leq m \leq (1+\beta)x} c(m) - c_f \beta x \right|^2 dx \ll_\varepsilon X^{1+2\sigma_0+\varepsilon} \int_{-\infty}^{\infty} |L(\sigma + it)|^2 \min \left\{ \beta^2, \frac{1}{1+|t|^2} \right\} dt.$$

Using Generalised Lindelöf Hypothesis assumption we may bound the right-hand side by  $\ll_\varepsilon X^{1+2\sigma_0+\varepsilon} \beta^{1-\varepsilon}$ . Plugging this into (5.5) yields

$$\int_X^{2X} \left| \sum_{x \leq m \leq x+H} c(m) - c_f \cdot H \right|^2 dx \ll_\varepsilon \frac{X}{H} \cdot X^{1+2\sigma_0+\varepsilon} \int_0^{8H/X} \beta^{1-\varepsilon} d\beta \ll_\varepsilon H X^{2\sigma_0+\varepsilon}.$$

For part (1) we have  $c(m) = |A(m)|^2$ , in which case  $\sigma_0 = 1/2 + \vartheta$  and  $c_f = r_f H_f(1)$ . For part (2) we have  $c(m) = A(m)$ ,  $\sigma_0 = 1/2$ , and  $c_f = 0$ . This completes the proof.  $\square$

The following bounds are also standard.

**Lemma 5.3.** *Let  $f$  be a Hecke–Maass cusp form for the group  $\mathrm{SL}_n(\mathbb{Z})$ . Assume Generalised Lindelöf Hypothesis for  $L(s, f)$  and  $L(s, f \times f)$  in the  $t$ -aspect. Let  $r_f$  be as in (5.1). Then we have*

(1)

$$\sum_{m \leq x} A(m) \ll_{\varepsilon} x^{1/2+\varepsilon};$$

(2)

$$\sum_{m \leq x} |A(m)|^2 = r_f H_f(1)x + O_{f,\varepsilon} \left( x^{1/2+\vartheta+\varepsilon} \right).$$

*Proof.* Again the underlying mechanism in both of the estimates is similar. Let  $c(m)$  and  $L(s)$  be as in the proof of Lemma 5.2. Suppose further that  $|c(m)| \ll_{\varepsilon} m^{\theta+\varepsilon}$  for some  $0 \leq \theta < 1$ . To estimate the coefficient sum

$$\sum_{m \leq x} c(m)$$

we study its first order Riesz weighted sum

$$R(x) := \sum_{m \leq x} c(m)(x-m).$$

By Mellin inversion we have

$$R(x) = \frac{1}{2\pi i} \int_{(2)} L(s) \frac{x^{s+1}}{s(s+1)} ds.$$

We shift the contour to the line  $\operatorname{Re}(s) = \sigma_0 + \varepsilon$  with a possible simple pole at  $s = 1$  contributing the residue  $c_f \cdot x^2/2$ . Thus

$$R(x) = c_f \cdot \frac{x^2}{2} + \frac{1}{2\pi i} \int_{(\sigma_0+\varepsilon)} L(s) \frac{x^{s+1}}{s(s+1)} ds.$$

Now observe that we have

$$\sum_{m \leq x} c(m) = R(x+1) - R(x) + O_{\varepsilon}(x^{\theta+\varepsilon})$$

and consequently

$$\sum_{m \leq x} c(m) = c_f \left( x + \frac{1}{2} \right) + \frac{1}{2\pi i} \int_{(\sigma_0+\varepsilon)} L(s) K_x(s) ds + O_{\varepsilon}(x^{\theta+\varepsilon}),$$

where

$$K_x(s) := \frac{(x+1)^{s+1} - x^{s+1}}{s(s+1)}.$$

We have the simple estimate

$$|K_x(s)| \ll \min \left\{ \frac{x^{\sigma}}{1+|t|}, \frac{x^{\sigma+1}}{1+|t|^2} \right\}.$$

Using the bounds to treat the ranges  $|t| \leq x$  and  $|t| > x$  separately, together with the bound  $L(\sigma_0 + \varepsilon + it) \ll (1+|t|)^{\varepsilon}$ , in the integral over  $\operatorname{Re}(s) = \sigma_0 + \varepsilon$  it follows that

$$\sum_{m \leq x} c(m) = c_f \left( x + \frac{1}{2} \right) + O_{\varepsilon} \left( x^{\max\{\sigma_0, \theta\} + \varepsilon} \right).$$

To prove (1), we have  $c(m) = A(m)$ ,  $c_f = 0$ ,  $\theta = \vartheta < 1/2$ , and  $\sigma_0 = 1/2$ . Thus we immediately obtain

$$\sum_{m \leq x} A(m) \ll_{\varepsilon} x^{1/2+\varepsilon},$$

as claimed.

For (2), we have  $c(m) = |A(m)|^2$ . By [6] we have  $L(s) = L(s, f \times f) H_f(s)$ , where  $H_f(s)$  is bounded and holomorphic in  $\operatorname{Re}(s) > 1/2 + \vartheta$ . Furthermore,  $L_f(s)$  has a simple pole at  $s = 1$  (from the pole of the Rankin–Selberg  $L$ -function at  $s = 1$ ) with residue

$$c_f = H_f(1) \operatorname{Res}_{s=1} L(s, f \times f) = r_f H_f(1).$$

We also have  $\sigma_0 = 1/2 + \vartheta$  and  $\theta = 2\vartheta < 1/2 + \vartheta$ . So have shown that

$$\sum_{m \leq x} |A(m)|^2 = r_f H_f(1)x + O_{f,\varepsilon} \left( x^{1/2+\vartheta+\varepsilon} \right),$$

as desired. This completes the proof.  $\square$

**5.3. Multiplicative functions.** We now gather results from the theory of multiplicative functions needed in the proofs. The following is [26, Corollary 1.1].

**Proposition 5.4.** *Let  $\mathcal{N}$  be a multiplicative subset of  $\mathbb{N}$ . Let  $\phi : \mathbb{N} \rightarrow [-1, 1]$  be a multiplicative function. Suppose that there exists a constant  $\alpha > 0$  such that for all  $2 \leq z \leq w$ ,*

$$(5.6) \quad \sum_{\substack{z < p \leq w \\ p \in \mathcal{N}}} \frac{1}{p} \geq \alpha \sum_{z < p \leq w} \frac{1}{p} - O \left( \frac{1}{\log z} \right).$$

*Then there exists a constant  $\kappa = \kappa(\alpha) > 0$  such that, for all  $\delta \in (0, 1/1000)$  and  $2 \leq h_0 \leq X$ ,*

$$\left| \frac{1}{h_0} \sum_{\substack{x \leq m \leq x+h_0\delta(\mathcal{N};X)^{-1} \\ m \in \mathcal{N}}} \phi(m) - \frac{1}{X\delta(\mathcal{N};X)} \sum_{\substack{m \sim X \\ m \in \mathcal{N}}} \phi(m) \right| < \delta$$

*outside of a set of  $[X, 2X]$  of cardinality  $\ll Xh_0^{-\delta^\kappa}$ . Here*

$$\delta(\mathcal{N}; X) := \prod_{\substack{p \leq X \\ p \notin \mathcal{N}}} \left( 1 - \frac{1}{p} \right).$$

We also require a version of Wirsing's theorem for non-negative multiplicative functions [34].

**Proposition 5.5.** *Let  $\phi : \mathbb{N} \rightarrow [0, \infty)$  be a multiplicative function so that*

- (1)  *$\sup_p |\phi(p)| \leq A$  for some  $A > 0$ ;*
- (2) *there exists  $\tau > 0$  so that*

$$\sum_{p \leq x} \frac{\phi(p) \log p}{p} = (\tau + o(1)) \log x;$$

(3)

$$\sum_p \sum_{v \geq 2} \frac{\phi(p^v)}{p^v} < \infty.$$

*Then we have*

$$\sum_{m \leq x} \phi(m) = \left( \frac{e^{-\gamma\tau}}{\Gamma(\tau)} + o(1) \right) x \prod_{p \leq x} \left( 1 - \frac{1}{p} \right) \sum_{j=0}^{\infty} \frac{\phi(p^j)}{p^j}.$$

We also require a result complementing Wirsing's theorem that allows us to say that if  $\phi$  is a multiplicative function such that  $|\phi(m)|$  satisfies the hypotheses of the previous proposition, then, provided  $\phi$  oscillates sufficiently, we have  $\sum_{m \leq x} \phi(m) = o(\sum_{m \leq x} |\phi(m)|)$ . The following result of this kind is due to Tenenbaum [33].

**Proposition 5.6.** *Let  $T \geq 1$  and  $\phi : \mathbb{N} \rightarrow \mathbb{C}$  be a multiplicative function such that*

- (1)  *$\sup_p |\phi(p)| \leq A$  for some  $A > 0$ ;*
- (2)

$$\sum_p \sum_{\nu \geq 2} \frac{|\phi(p^\nu)| (\log p^\nu)^2}{p^\nu} \leq B$$

*for some  $B > 0$ ;*

(3) Assume furthermore that there is a constant  $\beta > 0$  such that for any  $2 \leq y \leq x$  we have

$$\sum_{y \leq p \leq x} \frac{|\phi(p)|}{p} \geq \beta \log \left( \frac{\log x}{\log y} \right) + O(1).$$

Then, as  $x \rightarrow \infty$ ,

$$\left| \frac{1}{x} \sum_{m \leq x} \phi(m) \right| \ll_{A,B,\beta} \left( \frac{1}{x} \sum_{m \leq x} |\phi(m)| \right) \left( \frac{1 + m_\phi(x; T)}{e^{m_\phi(x; T)}} + \frac{1}{\sqrt{T}} + \frac{1}{\log x} \right),$$

where we have denoted

$$m_\phi(x; T) := \min_{|t| \leq T} \sum_{p \leq x} \frac{|\phi(p)| - \operatorname{Re}(\phi(p)p^{-it})}{p}.$$

To estimate the quantity  $m_\phi(x; T)$  we will require the following special case of [26, Lemma 5.1].

**Lemma 5.7.** *Let  $\phi : \mathbb{N} \rightarrow [-1, 1]$  be a real-valued multiplicative function that satisfies the estimate*

$$\sum_{z < p \leq w} \frac{|\phi(p)|}{p} \geq \alpha \sum_{z < p \leq w} \frac{1}{p} - O\left(\frac{1}{\log z}\right)$$

uniformly in  $2 \leq z < w$  for some  $\alpha > 0$ . Let  $\rho_\alpha := (\alpha/3) - (2/3\pi) \sin(\pi\alpha/2)$ . Then for any  $|t| \leq X$  and any  $0 < \rho < \rho_\alpha$  we have

$$\sum_{p \leq x} \frac{|\phi(p)| - \operatorname{Re}(\phi(p)p^{-it})}{p} \geq \rho \min\{\log \log x, 3 \log(1 + |t| \log x)\}.$$

The final tool from the theory of multiplicative functions is Shiu's bound [32].

**Lemma 5.8.** *Let  $\phi : \mathbb{N} \rightarrow \mathbb{C}$  be a multiplicative function satisfying  $|\phi(m)| \leq d_B(m)^C$  for all  $m \leq X$ . Let  $\sqrt{X} \leq Y \leq X$ ,  $\delta \in (0, 1)$ , and let  $Y^\delta \leq y \leq Y$ . Then*

$$\sum_{Y \leq m \leq Y+y} |\phi(m)| \ll_{B,C,\delta} y \mathcal{P}_\phi(X),$$

where

$$\mathcal{P}_\phi(X) := \prod_{p \leq X} \left( 1 + \frac{|\phi(p)| - 1}{p} \right).$$

## 6. PROOF OF THEOREM 1.2

Put  $g(m) := \operatorname{sgn}(A_f(m))$  and  $r(m) := |g(m)| = 1_{A_f(m) \neq 0}$ . Clearly both  $g$  and  $r$  are multiplicative functions, and both vanish if  $A_f(m) = 0$ . Let

$$\mathcal{R}_f(x; H) := \sum_{x \leq m \leq x+H} r(m) = \sum_{\substack{x \leq m \leq x+H \\ A_f(m) \neq 0}} 1 \quad \text{and} \quad \mathcal{G}_f(x; H) := \sum_{x \leq m \leq x+H} g(m) = \sum_{x \leq m \leq x+H} \operatorname{sgn}(A_f(m)).$$

The goal is to relate these short averages to averages over dyadic intervals  $[X, 2X]$ . Set  $h_0 := H \Delta_f(X)$  and let  $0 < \eta < 1/1000$  be a parameter chosen later. Applying Proposition 5.4 with the choices  $\phi = 1$  and  $\phi = g$  (with  $\mathcal{N} = \{m \in \mathbb{N} : A_f(m) \neq 0\}$ ) gives

$$(6.1) \quad \mathcal{R}_f(X, H) = \frac{H}{X} \sum_{m \sim X} r(m) + O(\eta H \Delta_f(X))$$

and

$$(6.2) \quad \mathcal{G}_f(X, H) = \frac{H}{X} \sum_{m \sim X} g(m) + O(\eta H \Delta_f(X))$$

simultaneously outside of a subset of  $x \in [X, 2X]$  with size  $\ll X h_0^{-\eta^\kappa}$  for some absolute  $\kappa > 0$ . To see that the condition (5.6) holds in this context, note that under Generalised Ramanujan–Petersson Conjecture we have a simple bound

$$1_{\{A_f(p) \neq 0\}} \geq \frac{|A_f(p)|^2}{n^2}.$$

Using this together with (5.2) gives the estimate

$$(6.3) \quad \sum_{\substack{z < p \leq w \\ A_f(p) \neq 0}} \frac{1}{p} \geq \frac{1}{n^2} \sum_{z < p \leq w} \frac{1}{p} - O_f\left(\frac{1}{\log z}\right)$$

needed.

Choose  $\eta = (\log h_0)^{-1/2\kappa}$ . Note that  $0 < \eta < 1/1000$  for sufficiently large  $X$  as  $h_0 \rightarrow \infty$  along with  $X$ . With this choice we have that  $h_0^{-\eta^\kappa} = \exp(-\sqrt{\log h_0})$ . Thus the size of the exceptional set is  $\ll_f X \exp(-\sqrt{\log h_0}) = o_f(X)$ . Now to complete the proof it suffices to show that, under Generalised Ramanujan–Petersson Conjecture,

(1)

$$\sum_{m \sim X} r(m) \asymp_f X \Delta_f(X);$$

(2)

$$\sum_{m \sim X} g(m) = o_f(X \Delta_f(X)).$$

Indeed, under these estimates it follows from (6.1) and (6.2) that outside of a subset of  $[X, 2X]$  of size  $o_f(X)$  we simultaneously have

$$\mathcal{R}_f(x; H) \asymp_f H \Delta_f(X) \quad \text{and} \quad \mathcal{G}_f(x; H) = o_f(H \Delta_f(X)).$$

This finishes the proof as

$$\mathcal{N}_f^\pm(x + H) - \mathcal{N}_f^\pm(x) = \frac{1}{2} (\mathcal{R}_f(x; H) \pm \mathcal{G}_f(x; H)).$$

To see that (1) holds, by Proposition 5.5 and a straightforward computation we have

$$\sum_{m \sim X} r(m) \asymp_f X \prod_{p \leq X} \left(1 + \frac{r(p) - 1}{p}\right)$$

as  $r(p) \in \{0, 1\}$ .

But

$$r(p) - 1 = \begin{cases} 0 & \text{if } A_f(p) \neq 0 \\ -1 & \text{if } A_f(p) = 0 \end{cases}$$

and so

$$\prod_{p \leq X} \left(1 + \frac{r(p) - 1}{p}\right) = \Delta_f(X).$$

This gives (1).

The argument to establish (2) is slightly more elaborate. For an application of Proposition 5.6 we require a lower bound for the quantity  $m_g(X; T)$ . For this we need to obtain a lower bound for

$$\sum_{\substack{z < p \leq w \\ A_f(p) < 0}} \frac{1}{p}$$

uniformly in  $2 \leq z < w$ .

As the underlying form satisfies the Ramanujan–Petersson Conjecture we have that  $|A_f(p)| \leq n$  and so  $|A_f(p)| \geq |A_f(p)|^2/n$ . Consequently, using (5.2),

$$(6.4) \quad \begin{aligned} \sum_{z < p \leq w} \frac{|A_f(p)|}{p} &\geq \frac{1}{n} \sum_{z < p \leq w} \frac{|A_f(p)|^2}{p} \\ &= \frac{1}{n} \sum_{z < p \leq w} \frac{1}{p} + O\left(\frac{1}{\log z}\right). \end{aligned}$$

Thus, we have

$$\begin{aligned} \sum_{\substack{z < p \leq w \\ A_f(p) > 0}} \frac{A_f(p)}{p} + \sum_{\substack{z < p \leq w \\ A_f(p) < 0}} \frac{|A_f(p)|}{p} &= \sum_{z < p \leq w} \frac{|A_f(p)|}{p} \\ &\geq \frac{1}{n} \sum_{z < p \leq w} \frac{1}{p} - O_f\left(\frac{1}{\log z}\right). \end{aligned}$$

Using (5.3) we have

$$\sum_{\substack{z < p \leq w \\ A_f(p) > 0}} \frac{A_f(p)}{p} - \sum_{\substack{z < p \leq w \\ A_f(p) < 0}} \frac{|A_f(p)|}{p} = O_f\left(\frac{1}{\log z}\right).$$

Combining these estimates we get

$$\min \left\{ \sum_{\substack{z < p \leq w \\ A_f(p) > 0}} \frac{A_f(p)}{p}, \sum_{\substack{z < p \leq w \\ A_f(p) < 0}} \frac{|A_f(p)|}{p} \right\} \geq \frac{1}{2n} \sum_{z < p \leq w} \frac{1}{p} - O_f\left(\frac{1}{\log z}\right).$$

As  $|A_f(p)| \leq n$  under Generalised Ramanujan–Petersson Conjecture, we conclude the estimates

$$(6.5) \quad \sum_{\substack{z < p \leq w \\ A_f(p) > 0}} \frac{1}{p} \geq \frac{1}{2n^2} \sum_{z < p \leq w} \frac{1}{p} - O_f\left(\frac{1}{\log z}\right)$$

and

$$(6.6) \quad \sum_{\substack{z < p \leq w \\ A_f(p) < 0}} \frac{1}{p} \geq \frac{1}{2n^2} \sum_{z < p \leq w} \frac{1}{p} - O_f\left(\frac{1}{\log z}\right)$$

uniformly for  $2 \leq z < w$ .

For  $T \geq 1$  recall that we have defined

$$m_g(X; T) = \min_{|t| \leq T} \sum_{p \leq X} \frac{|g(p)| - \operatorname{Re}(g(p)p^{-it})}{p}.$$

We claim that  $m_g(X; T) \gg \log \log X$  uniformly in  $0 < T \leq X$ . This can be deduced in an elementary manner using Lemma 5.7. Indeed, first observe, as  $g$  is real-valued, that we have

$$|g(p)| - \operatorname{Re}(g(p)p^{-it}) = \begin{cases} 1 - \cos(t \log p) & \text{if } A_f(p) > 0 \\ 1 + \cos(t \log p) & \text{if } A_f(p) < 0 \\ 0 & \text{if } A_f(p) = 0. \end{cases}$$

In the range  $(\log X)^{-1/2} \leq |t| \leq X$  we trivially have  $3 \log(1 + |t| \log X) \geq (3/2 + o(1)) \log \log X$ , and the claimed lower bound follows from Lemma 5.7 (which is applicable by (6.5) and (6.6)) at once. To treat the range  $|t| < (\log X)^{-1/2}$  we define the parameter  $Y := \exp(\sqrt{\log X}/4) < X$  and observe that for every

$p \leq Y$  we have  $|t| \log p \leq 1/4$ . Thus, for any such prime we have  $1 + \cos(t \log p) \geq 1 + \cos(-1/4) > 0$ . Now estimating trivially using non-negativity we have

$$\begin{aligned} \sum_{p \leq X} \frac{|g(p)| - \operatorname{Re}(g(p)p^{-it})}{p} &\geq \sum_{\substack{p \leq Y \\ A_f(p) < 0}} \frac{|g(p)| - \operatorname{Re}(g(p)p^{-it})}{p} \\ &\geq \left(1 + \cos\left(-\frac{1}{4}\right)\right) \sum_{\substack{p \leq Y \\ A_f(p) < 0}} \frac{1}{p}. \end{aligned}$$

We again obtain the required lower bound using (6.6) and observing that  $\log \log Y \asymp \log \log X$ . This completes the proof of the estimate  $m_g(X; T) \gg \log \log X$ .

Next we apply Tenenbaum's result (Proposition 5.6) with the choice  $T = (\log X)^4$ . Condition (3) in Proposition 5.6 is fulfilled by (6.3). The other two conditions hold obviously as  $g(m) \in \{0, 1\}$ . Using the previous observations, and the trivial upper bound  $m_g(X; T) \ll \log \log X$ , we have the estimate

$$\frac{1 + m_g(X; T)}{e^{m_g(X; T)}} \ll \frac{\log \log X}{e^{c_n \log \log X}}$$

for some constant  $c_n$  depending only on  $n$ . Thus

$$\begin{aligned} \left| \sum_{m \leq X} g(m) \right| &\ll \left( \sum_{m \leq X} r(m) \right) \left( \frac{\log \log X}{(\log X)^{c_n}} + \frac{1}{\log X} \right) \\ &= o \left( \sum_{m \leq X} r(m) \right). \end{aligned}$$

Given (1), this completes the proof of (2), and thus also the proof of Theorem 1.2.  $\square$

We finish this section by showing that Generalised Ramanujan–Petersson Conjecture implies  $\Delta_f(X) \gg_f (\log X)^{-1+1/n^2}$ . Indeed, from (6.3) we have

$$\sum_{\substack{p \leq X \\ A_f(p)=0}} \frac{1}{p} \leq \left(1 - \frac{1}{n^2}\right) \log \log X + O_f(1).$$

Therefore

$$\begin{aligned} \log \Delta_f(X) &= \sum_{\substack{p \leq X \\ A_f(p)=0}} \log \left(1 - \frac{1}{p}\right) \\ &= - \sum_{\substack{p \leq X \\ A_f(p)=0}} \frac{1}{p} + O(1) \\ &\geq - \left(1 - \frac{1}{n^2}\right) \log \log X + O_f(1), \end{aligned}$$

which gives the desired estimate

$$\Delta_f(X) \gg_f (\log X)^{-1+1/n^2}.$$

## 7. PROOF OF THEOREM 1.3

As in the proof of Theorem 1.2, it suffices to show that

$$\sum_{m \sim X} r(m) \asymp_f X \Delta_f(X) \quad \text{and} \quad \sum_{m \sim X} g(m) = o_f(X \Delta_f(X)),$$

where  $g(m)$  and  $r(m)$  are as in that proof.

Again the latter bound follows from Proposition 5.6 provided that the function  $g$  is sufficiently non-pretentious.

In the proof of Theorem 1.2 Generalised Ramanujan–Petersson Conjecture was used to verify conditions needed for applying Propositions 5.4, 5.5, 5.6, and estimating the quantity  $m_g(X; T)$ . We begin with a general result showing that these conditions can be verified with slightly weaker assumptions and afterwards justify them for  $\mathrm{GL}_2$  and  $\mathrm{GL}_3$  Hecke–Maass cusp forms unconditionally.

Let us assume that the following estimates hold uniformly in  $2 \leq z < w$ :

$$(7.1) \quad \sum_{z < p \leq w} \frac{|A_f(p)|^2}{p} = \sum_{z < p \leq w} \frac{1}{p} + O_f\left(\frac{1}{\log z}\right);$$

$$(7.2) \quad \sum_{z < p \leq w} \frac{A_f(p)}{p} = O_f\left(\frac{1}{\log z}\right);$$

$$(7.3) \quad \sum_{z < p \leq w} \frac{|A_f(p)|^4}{p} \leq C_4 \sum_{z < p \leq w} \frac{1}{p} + O_f\left(\frac{1}{\log z}\right)$$

for some absolute constant  $C_4 > 0$ .

By the Cauchy–Schwarz inequality we have

$$\left( \sum_{z < p \leq w} \frac{|A_f(p)|^2}{p} \right)^2 \leq \left( \sum_{\substack{z < p \leq w \\ A_f(p) \neq 0}} \frac{1}{p} \right) \left( \sum_{z < p \leq w} \frac{|A_f(p)|^4}{p} \right).$$

Combining this with the assumptions (7.1) and (7.3) gives

$$(7.4) \quad \sum_{\substack{z < p \leq w \\ A_f(p) \neq 0}} \frac{1}{p} \geq \frac{1}{C_4} \sum_{z < p \leq w} \frac{1}{p} - O_f\left(\frac{1}{\log z}\right).$$

As noted in [26, (13)], such an estimate implies that

$$\sum_{m \sim X} r(m) \asymp_{f, C_4} X \Delta_f(X)$$

with  $r(m) = 1_{A_f(m) \neq 0}$ . Note also that arguing as in the end of Section 6, (7.4) leads to the bound  $\Delta_f(X) \gg_f (\log X)^{-1+1/C_4}$ . Now it is enough to show that the conditions (7.1)–(7.3) imply

$$(7.5) \quad \sum_{m \sim X} g(m) = o_f(X \Delta_f(X)).$$

We need to verify that both inequalities  $A_f(p) > 0$  and  $A_f(p) < 0$  occur with sufficiently many primes  $p$ . By interpolation we have

$$\left( \sum_{z < p \leq w} \frac{|A_f(p)|^2}{p} \right)^3 \leq \left( \sum_{z < p \leq w} \frac{|A_f(p)|}{p} \right)^2 \left( \sum_{z < p \leq w} \frac{|A_f(p)|^4}{p} \right).$$

Combining this with the assumptions (7.1) and (7.3) gives

$$\sum_{z < p \leq w} \frac{|A_f(p)|}{p} \geq \frac{1}{\sqrt{C_4}} \sum_{z < p \leq w} \frac{1}{p} - O_f\left(\frac{1}{\log z}\right).$$

Using the assumption (7.2) it follows that

$$(7.6) \quad \sum_{\substack{z < p \leq w \\ A_f(p) < 0}} \frac{|A_f(p)|}{p} \geq \frac{1}{2\sqrt{C_4}} \sum_{z < p \leq w} \frac{1}{p} - O_f\left(\frac{1}{\log z}\right)$$

and a similar inequality holds for sum over the primes with  $A(p) > 0$ .

By the Cauchy–Schwarz inequality we have

$$\left( \sum_{\substack{z < p \leq w \\ A_f(p) < 0}} \frac{|A_f(p)|}{p} \right)^2 \leq \left( \sum_{\substack{z < p \leq w \\ A_f(p) < 0}} \frac{1}{p} \right) \left( \sum_{z < p \leq w} \frac{|A_f(p)|^2}{p} \right)$$

from which we deduce, using (7.1) and (7.6),

$$\sum_{\substack{z < p \leq w \\ A_f(p) < 0}} \frac{1}{p} \geq \left( \frac{1}{4C_4} + o(1) \right) \sum_{z < p \leq w} \frac{1}{p}.$$

Now the argument can be completed as in the proof of Theorem 1.2. Indeed, the estimate above yields  $m_g(X; T) \gg \log \log X$  uniformly in  $0 < T \leq X$ . By Proposition 5.6 this leads to (7.5).

So to complete the proof it suffices to verify conditions (7.1)–(7.3) unconditionally for self-dual  $\mathrm{GL}_2$  and  $\mathrm{GL}_3$  Hecke–Maass cusp forms. Let us start by establishing part (1) of the theorem. Let  $\mathrm{Sym}^2 f$  be the symmetric square lift of  $f$ , which is a  $\mathrm{GL}_3$  Hecke–Maass cusp form due to a result of Gelbart and Jacquet [2]. The Fourier coefficients of  $f$  and  $\mathrm{Sym}^2 f$  at primes are connected by the relation  $A_f(p)^2 = A_{\mathrm{Sym}^2 f}(p) + 1$  and so in particular

$$A_f(p)^4 = A_{\mathrm{Sym}^2 f}(p)^2 + 2A_{\mathrm{Sym}^2 f}(p) + 1.$$

Using Lemma 5.1 with  $\pi = \pi' = \mathrm{Sym}^2 f$  we have

$$\sum_{p \leq x} A_{\mathrm{Sym}^2 f}(p)^2 \log p = x + O\left(xe^{-c\sqrt{\log x}}\right).$$

Applying the same lemma with  $\pi = \mathrm{Sym}^2 f$  and  $\pi' = 1$  (the trivial representation of  $\mathrm{GL}_1$ ) gives

$$\sum_{p \leq x} A_{\mathrm{Sym}^2 f}(p) \log p \ll xe^{-c\sqrt{\log x}}.$$

Combining these observations and the usual prime number theorem gives

$$\begin{aligned} \sum_{p \leq x} A_f(p)^4 \log p &= \sum_{p \leq x} A_{\mathrm{Sym}^2 f}(p)^2 \log p + 2 \sum_{p \leq x} A_{\mathrm{Sym}^2 f}(p) \log p + \sum_{p \leq x} \log p \\ &= 2x + O\left(xe^{-c\sqrt{\log x}}\right). \end{aligned}$$

This gives (7.3) with  $C_4 = 2$  by partial summation. We conclude using the bound  $\Delta_f(X) \gg (\log X)^{-1+1/C_4}$ .

The proof of part (2) requires more elaborate arguments. Let  $\Pi$  be a fixed self-dual unitary cuspidal automorphic representation of  $\mathrm{GL}_3(\mathbb{A}_{\mathbb{Q}})$  with normalised Fourier coefficients  $A_{\Pi}(m) := A(m, 1)$ . As before, put

$$\Delta_{\Pi}(X) := \prod_{\substack{p \leq X \\ A_{\Pi}(p)=0}} \left(1 - \frac{1}{p}\right).$$

We will use a result of Ramakrishnan [28, Theorem A] saying that every self-dual cuspidal automorphic representation of  $\mathrm{GL}_3(\mathbb{A}_{\mathbb{Q}})$  is, up to a quadratic twist, an adjoint/symmetric square lift from  $\mathrm{GL}_2$ . So, there exists a non-dihedral cuspidal representation  $\sigma$  of  $\mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}})$  and a quadratic character  $\nu$  (i.e.  $\nu^2 = 1$ ) such that

$$\Pi \simeq \mathrm{Ad}(\sigma) \otimes \nu,$$

where  $\mathrm{Ad}(\sigma) := \mathrm{Sym}^2 \sigma \otimes \omega_{\sigma}^{-1}$  with  $\omega_{\sigma}$  being the central character of  $\sigma$ .

We distinguish into cases depending on whether  $\sigma$  is neither tetrahedral nor octahedral or not. Assume first that  $\sigma$  is of cyclic or icosahedral type. For such representations we show the following.

**Proposition 7.1.** *Assume that  $\Pi \simeq \text{Ad}(\sigma) \otimes \nu$ , where  $\sigma$  is a cuspidal automorphic representation of  $\text{GL}_2(\mathbb{A}_{\mathbb{Q}})$  that is either of cyclic or icosahedral type, and  $\nu$  is a quadratic character. Then we have uniformly in  $2 \leq z < w$  that*

(1)

$$\sum_{z < p \leq w} \frac{|A_{\Pi}(p)|^2}{p} = \sum_{z < p \leq w} \frac{1}{p} + O_{\Pi} \left( \frac{1}{\log z} \right);$$

(2)

$$\sum_{z < p \leq w} \frac{A_{\Pi}(p)}{p} = O_{\Pi} \left( \frac{1}{\log z} \right);$$

(3)

$$\sum_{z < p \leq w} \frac{|A_{\Pi}(p)|^4}{p} = 3 \sum_{z < p \leq w} \frac{1}{p} + O_{\Pi} \left( \frac{1}{\log z} \right).$$

*Proof.* The first two bounds are direct consequences of Lemma 5.1 and partial summation. For (3) we argue as follows. For simplicity, put  $\rho := \text{Ad}(\sigma)$  and  $\tau := \text{Sym}^4 \sigma \otimes \omega_{\sigma}^{-2}$ . By Kim's symmetric fourth power functoriality [11],  $\tau$  is an automorphic representation for the group  $\text{GL}_5(\mathbb{A}_{\mathbb{Q}})$ . Furthermore, by the result of Kim and Shahidi [13],  $\text{Sym}^4 \sigma$  is cuspidal as we assumed  $\sigma$  to be of cyclic or icosahedral type.

Denote the Satake parameters of  $\sigma$  at a prime  $p$  as  $\alpha_p$  and  $\beta_p$ . Then we have the relations

$$A_{\rho}(p) = \alpha_p \beta_p^{-1} + \alpha_p^{-1} \beta_p + 1$$

and

$$A_{\tau}(p) = \alpha_p^2 \beta_p^{-2} + \alpha_p \beta_p^{-1} + 1 + \alpha_p^{-1} \beta_p + \alpha_p^{-2} \beta_p^2.$$

Thus a direct calculation gives

$$\begin{aligned} A_{\rho}(p)^2 &= \alpha_p^2 \beta_p^{-2} + \alpha_p^{-2} \beta_p^2 + 2\alpha_p \beta_p^{-1} + 2\alpha_p^{-1} \beta_p + 3 \\ &= A_{\tau}(p) + A_{\rho}(p) + 1. \end{aligned}$$

Squaring yields

$$A_{\rho}(p)^4 = A_{\tau}(p)^2 + A_{\rho}(p)^2 + 1 + 2A_{\tau}(p)A_{\rho}(p) + 2A_{\tau}(p) + 2A_{\rho}(p).$$

Noting that  $A_{\Pi}(p) = \nu(p)A_{\rho}(p)$  and  $\nu(p)^2 = 1$  we have

(7.7)

$$\begin{aligned} &\sum_{p \leq x} A_{\Pi}(p)^4 \log p \\ &= \sum_{p \leq x} A_{\tau}(p)^2 \log p + \sum_{p \leq x} A_{\rho}(p)^2 \log p + \sum_{p \leq x} \log p + 2 \sum_{p \leq x} A_{\tau}(p)A_{\rho}(p) \log p + 2 \sum_{p \leq x} A_{\tau}(p) \log p + 2 \sum_{p \leq x} A_{\rho}(p) \log p. \end{aligned}$$

Using Lemma 5.1 we have the estimates

$$\sum_{p \leq x} A_{\rho}(p)^2 \log p = x + O_{\Pi} \left( x e^{-c\sqrt{\log x}} \right) \quad \text{and} \quad \sum_{p \leq x} A_{\rho}(p) \log p \ll x e^{-c\sqrt{\log x}}.$$

For  $\tau$ , which is  $\text{GL}_5$  representation, one needs Hypothesis H in order to apply part (2) of Lemma 5.1. Fortunately this is exactly one of the special higher degree cases in which this hypothesis is known [12]. As  $\tau$  is cuspidal and self-dual, we have

$$\sum_{p \leq x} A_{\tau}(p)^2 \log p = x + O_{\sigma} \left( \frac{x}{\log x} \right)$$

by using part (2) of Lemma 5.1. Likewise, applying Lemma 5.1 to the pair  $(\pi, \pi') = (\tau, \rho)$  we have, as  $\tau$  and  $\rho$  clearly cannot be twist equivalent simply because they are of different degree,

$$\sum_{p \leq x} A_\tau(p) A_\rho(p) \log p = O_\sigma \left( \frac{x}{\log x} \right).$$

Similarly we have

$$\sum_{p \leq x} A_\tau(p) \log p = O_\sigma \left( \frac{x}{\log x} \right)$$

and finally by the ordinary prime number theorem

$$\sum_{p \leq x} \log p = x + O \left( x e^{-c\sqrt{\log x}} \right).$$

Using these in (7.7) gives

$$\sum_{p \leq x} |A_\Pi(p)|^4 \log p = 3x + O_\Pi \left( \frac{x}{\log x} \right),$$

from which (3) follows by partial summation. This completes the proof of the proposition.  $\square$

Thus conditions (7.1)-(7.3) hold for cyclic and icosahedral representations. Now the proof can be completed as before in the case that  $\sigma$  is either of these types.

In the opposite case ( $\sigma$  is tetrahedral or octahedral) again (7.1) and (7.2) hold by Lemma 5.1. However, for the fourth moment we need to proceed differently as  $\text{Sym}^4 \sigma$  is not cuspidal. But in this case Kim and Shahidi [13] have shown that  $\sigma$  satisfies Generalised Ramanujan–Pettersson Conjecture holds for all primes  $p$ . Thus using the estimate  $|A_\Pi(p)| \leq 3$  we trivially have that  $|A_\Pi(p)|^4 \leq 9|A_\Pi(p)|^2$ . Hence, using the second moment bound we deduce that

$$\begin{aligned} \sum_{z < p \leq w} \frac{|A_\Pi(p)|^4}{p} &\leq 9 \sum_{z < p \leq w} \frac{|A_\Pi(p)|^2}{p} \\ &\leq 9 \sum_{z < p \leq w} \frac{1}{p} + O_\Pi \left( \frac{1}{\log z} \right) \end{aligned}$$

uniformly in  $2 \leq z < w$ . So (7.3) holds also in this case. The proof is completed.  $\square$

Finally, we deduce Corollary 1.5. Let  $\mathcal{S}$  denote the set of points  $x \sim X$  for which the estimate

$$\mathcal{N}_f^\pm(x+H) - \mathcal{N}_f^\pm(x) \gg H \Delta_f(X)$$

holds. By Theorem 1.2 the set  $\mathcal{S}$  has cardinality  $(1 - o(1))X$ . Choose a maximal  $H$ -separated set  $x_1, \dots, x_J \in \mathcal{S}$ . Maximality implies that the intervals  $[x_j - H, x_j + H]$  cover the set  $\mathcal{S}$  and consequently  $J \gg X/H$ . Set  $I_j := [x_j - H/4, x_j]$  for  $j = 1, \dots, J$ . These intervals are pairwise disjoint and for  $y \in I_j$  we have  $(x_j, x_j + H/2] \subset (y, y + H]$ . Therefore every positive or negative coefficient that is counted in the interval  $(x_j, x_j + H/2]$  is also counted in the interval  $(y, y + H]$ . Applying Theorem 1.2 to the intervals  $(x_j, x_j + H/2]$  yields the desired result.

## 8. PROOFS OF THEOREMS 1.6 AND 1.7

**8.1. Proof of Theorem 1.6.** Let us set

$$\mathcal{P}_f := \{p \in \mathbb{P} : A_f(p) \neq 0\}.$$

As we assume Generalised Ramanujan–Pettersson Conjecture, we have  $|A_f(p)| \leq n$  and so

$$1_{\mathcal{P}_f}(p) \geq \frac{|A_f(p)|^2}{n^2}.$$

Thus, using the uniform Rankin–Selberg estimate (5.2) we have

$$(8.1) \quad \sum_{\substack{z < p \leq w \\ p \in \mathcal{P}_f}} \frac{1}{p} \geq \frac{1}{n^2} \sum_{z < p \leq w} \frac{|A_f(p)|^2}{p} \\ \geq \frac{1}{n^2} \sum_{z < p \leq w} \frac{1}{p} + O_f \left( \frac{1}{\log z} \right).$$

Using a method of Landau, Qu [27, Theorem 1.1] proved that the sequence  $\{A_f(m)\}_{m=1}^\infty$  has infinitely many sign changes, and so in particular some coefficient  $A_f(r)$  is negative. Define the multiplicative function

$$\ell(m) := \mu(m)^2 1_{p|m \Rightarrow p \in \mathcal{P}_f, (p,r)=1},$$

i.e.,  $\ell(m)$  is the characteristic function of squarefree integers composed of primes  $p$  coprime to  $r$  at which  $A_f(p) \neq 0$ .

We wish to apply Wirsing’s theorem (Proposition 5.5). Towards this we define a multiplicative function  $h$  by

$$h(p) := \begin{cases} \frac{|A_f(p)|^2}{n^2} & \text{if } (p, r) = 1 \\ 0 & \text{if } p|r \end{cases} \quad \text{and} \quad h(p^\nu) := 0 \quad \text{for } \nu \geq 2.$$

As we assume Generalised Ramanujan–Petersson Conjecture,  $0 \leq h(p) \leq 1$  and so  $0 \leq h(m) \leq \ell(m)$  for every  $m$ . Furthermore, using Lemma 5.1 and partial summation we have

$$\begin{aligned} \sum_{p \leq x} \frac{h(p) \log p}{p} &= \frac{1}{n^2} \sum_{\substack{p \leq x \\ (p,r)=1}} \frac{|A_f(p)|^2 \log p}{p} \\ &= \frac{1}{n^2} \sum_{p \leq x} \frac{|A_f(p)|^2 \log p}{p} + O_r(1) \\ &= \left( \frac{1}{n^2} + o(1) \right) \log x. \end{aligned}$$

Now Proposition 5.5 gives the lower bound

$$(8.2) \quad \sum_{m \leq x} \ell(m) \geq \sum_{m \leq x} h(m) \gg x \prod_{p \leq x} \left( 1 - \frac{1}{p} \right) \sum_{j=0}^\infty \frac{h(p^j)}{p^j} \gg x (\log x)^{1/n^2-1}.$$

Consider the set

$$\mathcal{A} := \{m \in \mathbb{N}, \text{ squarefree}, (m, r) = 1, p|m \Rightarrow p \in \mathcal{P}_f\}.$$

Next we observe that for  $m \in \mathcal{A}$  we have  $A_f(m) \neq 0$  and  $A_f(mr) = A_f(m)A_f(r)$ . At least half of the numbers  $\mathcal{A} \cap [1, x/r]$  are of the same sign (wlog positive). By the preceding observations there are also at least this number of negative coefficients in the larger set  $\mathcal{A} \cap [1, x]$  as  $\text{sgn}(A_f(m)A_f(mr)) = \text{sgn}(A_f(r)) < 0$ . As the cardinality of the set  $\mathcal{A} \cap [1, x]$  is  $\gg x(\log x)^{1/n^2-1}$  by (8.2), this concludes the proof.  $\square$

**8.2. Proof of Theorem 1.7.** We first prove a general result and then specialise to  $\text{GL}_n$  with  $n \leq 3$ . Let  $a : \mathbb{N} \rightarrow \mathbb{Z}$  be a multiplicative function and define a multiplicative set

$$\mathcal{M}_a := \{m \in \mathbb{N} : a(m) \neq 0\}.$$

Suppose that there exists an absolute constant  $\alpha > 0$  so that

$$(8.3) \quad \sum_{\substack{z < p \leq w \\ p \in \mathcal{M}_a}} \frac{1}{p} > \alpha \sum_{z < p \leq w} \frac{1}{p} - O \left( \frac{1}{\log z} \right)$$

uniformly in  $2 \leq z < w$ . Then we claim that

$$(8.4) \quad \# \{m \in \mathcal{M}_a \cap [X, 2X]\} \gg_\alpha X \Delta_a(X).$$

Indeed, this is a fairly direct consequence of [26, Corollary 1.2 (i)], which says that for a given  $2 \leq h_0 \leq X$  an interval  $[x, x + h_0 \Delta_a(X)^{-1}]$  of length  $h_0 \Delta_a(X)^{-1}$  contains  $\gg_\alpha h_0$  elements  $m \in \mathbb{N}$  such that  $a(m) \neq 0$  for all but  $\ll_{\alpha, \varepsilon} X h_0^{-1/2+\varepsilon}$  of  $x \sim X$ . Choosing  $h_0 = C_\alpha$  to be a sufficiently large fixed constant depending on  $\alpha$  it follows that for a positive proportion of  $x \sim X$  we have that

$$\#\{m \in \mathcal{M}_a \cap [x, x + C_\alpha \Delta_a(X)^{-1}]\} \gg_\alpha 1.$$

Summing over  $x \sim X$  and noting that every integer is counted at most  $O(\Delta_a(X)^{-1})$  times gives (8.4).

Let us now suppose that  $a(m)$  are either Hecke eigenvalues of a  $\mathrm{GL}_2$  or  $\mathrm{GL}_3$  Hecke–Maass cusp form  $f$ . Now choose  $r \geq 2$  for which  $A_f(r) < 0$ , which exists by a result of Qu [27]. We consider the set

$$\mathcal{M}_{f,r} := \{m \in \mathbb{N} : A_f(m) \neq 0, (m, r) = 1\}.$$

Note that condition (8.3) holds with  $\mathcal{M}_a$  replaced by  $\mathcal{M}_{f,r}$  as removing finite number of primes dividing  $r$  causes an error  $\ll 1/z$ . Thus we have

$$\#\{m \sim X : m \in \mathcal{M}_{f,r}\} \gg_{f,r} X \Delta_f(X).$$

For every  $m \in \mathcal{M}_{f,r}$  we have  $A_f(mr) = A_f(m)A_f(r)$  and so  $A_f(m)$  and  $A_f(mr)$  have opposite signs. Note that for  $m \in \mathcal{M}_{f,r}$  with  $m \leq X/r$  the pairs  $(m, mr)$  are disjoint as  $(m, r) = 1$ . Thus it follows that

$$\mathcal{N}_f^\pm(X) \gg_f X \Delta_f(X).$$

From the proof of Theorem 1.3 (see (7.4) and comments following it) we have that under the condition (8.3) it follows that  $\Delta_f(X) \gg (\log X)^{\alpha-1}$  and the condition holds with  $\alpha = 1/3$  for a self-dual  $\mathrm{GL}_3$  Maass cusp form  $f$  and  $\alpha = 1/2$  for  $\mathrm{GL}_2$  Maass cusp forms, as we have seen. This completes the proof.  $\square$

## 9. PROOFS OF THEOREMS 1.9 AND 1.10

**9.1. Proof of Theorem 1.9.** (1) Let us define the set

$$\mathcal{U} := \left\{ x \sim X : \sum_{x \leq m \leq x+H} |A(m)|^2 < \frac{r_f H_f(1) \cdot H}{2} \right\}.$$

Note that for  $x \in \mathcal{U}$  we have

$$\frac{r_f H_f(1) \cdot H}{2} < r_f H_f(1) \cdot H - \sum_{x \leq m \leq x+H} |A(m)|^2$$

and so from part (1) of Lemma 5.2 and Chebyshev's inequality it immediately follows that

$$|\mathcal{U}| \ll \frac{1}{H^2} \int_X^{2X} \left| \sum_{x \leq m \leq x+H} |A(m)|^2 - r_f H_f(1) \cdot H \right|^2 dx \ll_\varepsilon \frac{X^{1+2\vartheta+\varepsilon}}{H}.$$

Let us fix an arbitrarily small  $\varepsilon > 0$ . For  $x \in \mathcal{U}^c$  we clearly have

$$\sum_{x \leq m \leq x+H} |A(m)| \gg_\varepsilon H X^{-\vartheta-\varepsilon}$$

using the trivial estimate  $\max_{x \leq m \leq x+H} |A(m)| \ll_\varepsilon X^{\vartheta+\varepsilon}$ .

Define an another set

$$\mathcal{V} := \left\{ x \sim X : \left| \sum_{x \leq m \leq x+H} A(m) \right| > r_f H_f(1) \cdot H X^{-\vartheta-2\varepsilon} \right\}.$$

Then by part (2) of Lemma 5.2 and a simple application of Chebyshev's inequality we have

$$|\mathcal{V}| \ll \frac{X^{2\vartheta+4\varepsilon}}{H^2} \int_X^{2X} \left| \sum_{x \leq m \leq x+H} A(m) \right|^2 dx \ll \frac{X^{1+2\vartheta+\varepsilon}}{H}.$$

Thus for  $x \in \mathcal{U}^c \cap \mathcal{V}^c$  we have

$$\sum_{x \leq m \leq x+H} |A(m)| \pm \sum_{x \leq m \leq x+H} A(m) \gg_\varepsilon H X^{-\vartheta-\varepsilon},$$

and consequently

$$\sum_{\substack{x \leq m \leq x+H \\ A(m) \leq 0}} A(m) = \frac{1}{2} \left( \sum_{x \leq m \leq x+H} |A(m)| \pm \sum_{x \leq m \leq x+H} A(m) \right) \gg_\varepsilon H X^{-\vartheta-\varepsilon},$$

which gives the claim in view of the pointwise estimate  $|A(m)| \ll_\varepsilon m^{\vartheta+\varepsilon}$ . The exceptional set for which this does not hold has size  $\ll_\varepsilon X^{1+2\vartheta+\varepsilon}/H$  by the bounds on the sizes on  $\mathcal{U}$  and  $\mathcal{V}$ . This completes the proof of the first part.

(2) By Lemma 5.3 we have the estimates

$$\sum_{x \leq m \leq x+H} A(m) \ll_\varepsilon x^{1/2+\varepsilon} \quad \text{and} \quad \sum_{x \leq m \leq x+H} |A(m)|^2 = r_f H_f(1) \cdot H + O_{f,\varepsilon} \left( X^{1/2+\vartheta+\varepsilon} \right).$$

As  $H \geq X^{1/2+\vartheta+\eta}$  (for some fixed  $\eta > 0$ ), it follows that

$$\sum_{x \leq m \leq x+H} |A(m)|^2 \asymp H,$$

and consequently

$$\sum_{x \leq m \leq x+H} |A(m)| \gg_\varepsilon H X^{-\vartheta-\varepsilon},$$

using the pointwise bound  $|A(m)| \ll X^{\vartheta+\varepsilon}$ .

Furthermore, in the same range of  $H$  we clearly have

$$\sum_{x \leq m \leq x+H} A(m) = o \left( \sum_{x \leq m \leq x+H} |A(m)| \right).$$

Combining these estimates we deduce that

$$\sum_{x \leq m \leq x+H} |A(m)| \pm \sum_{x \leq m \leq x+H} A(m) \gg_\varepsilon H X^{-\vartheta-\varepsilon},$$

and the proof can be concluded as in part (1).  $\square$

**Remark 9.1.** One may wonder whether the results of this section can be improved using a bilinear structure as in [7]. Unfortunately, it seems that such a strategy does not yield an improvement without assuming the full Generalised Ramanujan–Petersson Conjecture.

**9.2. Proof of Theorem 1.10.** Fix  $q > 2$ . By Shiu’s bound (Lemma 5.8) we have

$$(9.1) \quad \sum_{x \leq m \leq x+H} |A(m)|^q \ll \frac{H}{\log x} \exp \left( \sum_{p \leq x} \frac{|A(p)|^q}{p} \right).$$

Under Generalised Ramanujan–Petersson Conjecture we have  $|A(p)|^q \leq n^{q-2} |A(p)|^2$ . Together with the Rankin–Selberg asymptotics (5.2) this gives

$$\sum_{p \leq x} \frac{|A(p)|^q}{p} \leq n^{q-2} \sum_{p \leq x} \frac{|A(p)|^2}{p} = n^{q-2} \log \log x + O(1).$$

Plugging this into (9.1) leads to

$$(9.2) \quad \sum_{x \leq m \leq x+H} |A(m)|^q \ll H (\log x)^{n^{q-2}-1}.$$

By simple interpolation we also have the estimate

$$\sum_{x \leq m \leq x+H} |A(m)|^2 \leq \left( \sum_{x \leq m \leq x+H} |A(m)| \right)^{\frac{q-2}{q-1}} \left( \sum_{x \leq m \leq x+H} |A(m)|^q \right)^{\frac{1}{q-1}}.$$

As seen in the proof of part (2) of the previous theorem, thanks to the assumption<sup>8</sup>  $H \geq x^{1/2+\eta}$  we have that the left-hand side of the interpolation inequality is  $\asymp H$ . Thus, recalling (9.2) we deduce the lower bound

$$\sum_{x \leq m \leq x+H} |A(m)| \gg H(\log x)^{-(n^{q-2}-1)/(q-2)}.$$

We also recall from proof of part (2) in the previous theorem that

$$\sum_{x \leq m \leq x+H} A(m) = o \left( \sum_{x \leq m \leq x+H} |A(m)| \right)$$

for  $H \geq x^{1/2+\eta}$ .

Thus

$$\sum_{x \leq m \leq x+H} |A(m)| \pm \sum_{x \leq m \leq x+H} A(m) \gg H(\log x)^{-(n^{q-2}-1)/(q-2)}.$$

By the Cauchy–Schwarz inequality we have

$$H^2(\log x)^{-2(n^{q-2}-1)/(q-2)} \ll \left( \mathcal{N}_f^\pm(x+H) - \mathcal{N}_f^\pm(x) \right) \sum_{x \leq m \leq x+H} |A(m)|^2.$$

This implies the claim by recalling the estimate

$$\sum_{x \leq m \leq x+H} |A(m)|^2 \asymp H$$

for  $H \geq x^{1/2+\eta}$  and choosing  $q$  to be arbitrarily close to 2 as well as noting that

$$\lim_{q \rightarrow 2} \frac{n^{q-2} - 1}{q - 2} = \log n.$$

The proof is completed. □

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<sup>8</sup>Recall that we assume Generalised Ramanujan–Petersson Conjecture here.

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