MASS EQUIDISTRIBUTION FOR SAITO-KUROKAWA LIFTS

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Abstract. Let F be a holomorphic cuspidal Hecke eigenform for $\mathrm{Sp}_4(\mathbb{Z})$ of weight k that is a Saito–Kurokawa lift. Assuming the Generalized Riemann Hypothesis (GRH), we prove that the mass of F equidistributes on the Siegel modular variety as $k \longrightarrow \infty$. As a corollary, we show under GRH that the zero divisors of Saito–Kurokawa lifts equidistribute as their weights tend to infinity.

1. Introduction

1.1. **Background.** A central problem in quantum chaos is to understand the distribution of mass of high energy Laplace-Beltrami eigenfunctions on a Riemannian manifold M. The fundamental Quantum Ergodicity Theorem of Shnirel'man [Shn74], Colin de Verdiére [CdV85] and Zelditch [Zel87] asserts that if the geodesic flow is ergodic on the unit cotangent bundle of M, then any sequence of eigenfunctions with eigenvalues tending to infinity contains a density one subsequence whose mass equidistributes. In the case that M is negatively curved, Rudnick and Sarnak [RS94] made the stronger conjecture that the quantum limit is unique, that is, for every sequence φ_ℓ of eigenfunctions with eigenvalues tending to infinity, the mass $|\varphi_\ell|^2$ equidistributes with respect to the normalized Liouville measure. This is known as the Quantum Unique Ergodicity (QUE) conjecture and in full generality is regarded as extremely difficult, despite some remarkable partial results [Ana08, AN07, DJ18, DJN22]. However, QUE has been proved for certain special arithmetic manifolds M which arise as quotients of symmetric spaces by arithmetic groups and have additional symmetries in the form of a large commuting family of Hecke operators [Lin06, Sou10, SV07, LR20, STS22].

Since Laplace–Beltrami eigenfunctions on arithmetic manifolds are instances of automorphic forms, one can consider variants of QUE by replacing the family of Laplace–Beltrami eigenfunctions with a suitable family of automorphic forms with certain parameters (e.g., weight, level, etc.) tending to infinity. Perhaps the most natural variant here is obtained by taking the family of holomorphic cusp forms of weight k (where we let $k \longrightarrow \infty$) on some fixed complex arithmetic manifold M. In the simplest rank 1 case that M equals the modular surface $\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}$, the corresponding mass equidistribution conjecture was first spelled out by Luo and Sarnak [LS03] and later proved by Holowinsky and Soundararajan [HS10] who combined a triple product L-function approach via Watson's formula [Wat08] with one based on shifted convolutions sums. This result, known as holomorphic QUE, has the beautiful corollary, proved by Rudnick [Rud05] that the zeros of all such Hecke cusp forms equidistribute. Holomorphic QUE on quotients of $\mathbb{H}$ by congruence subgroups (and more generally, quotients of $\mathbb{H}^m$ by congruence subgroups associated to a totally real number field of degree m) have now been established in various aspects [Mar11, Nel11, Nel12, NPS14, Hu18] by building upon the approach of Holowinsky–Soundararajan.

In this paper, we are interested in higher rank generalizations of holomorphic QUE. Precisely, let $\mathbb{H}_n$ denote the Siegel upper-half space of degree n and let $S_k(\mathrm{Sp}_{2n}(\mathbb{Z}))$ be the space of holomorphic Siegel cusp forms of weight k transforming with respect to the subgroup $\mathrm{Sp}_{2n}(\mathbb{Z}) \subset \mathrm{Sp}_{2n}(\mathbb{R})$. Let $\mathrm{d}\mu := (\det Y)^{-n-1}\mathrm{d}X\mathrm{d}Y$ be the usual $\mathrm{Sp}_{2n}(\mathbb{R})$-invariant measure on $\mathbb{H}_n$. The pushforward to

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$Y_n := \mathrm{Sp}_{2n}(\mathbb{Z})\backslash\mathbb{H}_n$ of the L^2-mass of $F \in S_k(\mathrm{Sp}_{2n}(\mathbb{Z}))$ is the finite measure given by

$$\mu_F(\phi) := \int_{Y_n} |F(Z)|^2 \phi(Z)\,(\det Y)^k \,\mathrm{d}\mu$$

for each bounded measurable function ϕ on Y_n, and let

$$D_F(\phi) := \frac{\mu_F(\phi)}{\mu_F(1)} - \frac{1}{\mathrm{vol}(Y_n)} \int_{Y_n} \phi(Z)\,\mathrm{d}\mu.$$

The quantity $D_F(\phi)$ compares the (normalized) measures attached to μ_F and μ against the test function ϕ. The following conjecture is the natural generalization of holomorphic QUE to higher rank.

Conjecture 1.1. *Fix a bounded continuous function ϕ on $\mathrm{Sp}_{2n}(\mathbb{Z})\backslash\mathbb{H}_n$. Let $F \in S_k(\mathrm{Sp}_{2n}(\mathbb{Z}))$ traverse a sequence of Hecke eigenforms. Then $D_F(\phi) \longrightarrow 0$ whenever $k \longrightarrow \infty$.*

The above conjecture (in a slightly different form) was first spelled out by Cogdell and Luo [CL11]. When $n = 1$, Conjecture 1.1 reduces to the holomorphic QUE conjecture mentioned above which was proved by Holowinsky and Soundararajan [HS10]. However, there has been very little progress in the direction of Conjecture 1.1 in the higher rank setting $n > 1$. To indicate the key difficulties, we note first that an analogue of Watson's formula [Wat08] is not known (nor expected) to exist if $n > 1$. Consequently, the direct relation between holomorphic QUE and the subconvexity problem in the classical case does not carry over to the setting of higher rank holomorphic Siegel cusp forms. Secondly, the unconditional techniques of Holowinsky-Soundararajan [HS10] are not directly applicable since they rely crucially on the multiplicativity of the coefficients, and the Fourier coefficients of Siegel cusp forms of higher rank $n > 1$ are highly non-multiplicative.[1]

1.2. **Results.** Due to the difficulty of Conjecture 1.1 in general, it seems reasonable to attempt it first for Siegel cusp forms that are *lifts* of some sort. Indeed, in the case $n = 1$, mass equidistribution was initially proved for Eisenstein series [LS95] and for dihedral/CM forms [Sar01, LY02] (which are both lifts from characters). For $n > 1$, the simplest lifts are the Saito–Kurokawa lifts, which exist for $n = 2$. The Saito-Kurokawa lifts can be explicitly constructed from classical half-integral weight forms via the theory of Jacobi forms [EZ85, §6]; they may also be viewed as lifts of classical integral weight forms thanks to the Shimura correspondence between half integral weight and integral weight forms. Furthermore, from the representation theoretic point of view, the Saito–Kurokawa lifts may be understood as a special case of Langlands functoriality realized via the theta correspondence [Sch05].

Our main result proves Conjecture 1.1 for Saito-Kurokawa lifts under the Generalized Riemann Hypothesis (GRH).

Theorem 1.2. *Assume GRH. Let $F \in S_k(\mathrm{Sp}_4(\mathbb{Z}))$ traverse a sequence of Hecke eigenforms that are Saito–Kurokawa lifts. Then, for each bounded continuous function ϕ on $\mathrm{Sp}_4(\mathbb{Z})\backslash\mathbb{H}_2$, we have $D_F(\phi) \longrightarrow 0$ whenever $k \longrightarrow \infty$. In other words,*

$$\frac{1}{\|F\|_2^2} \int_{\mathrm{Sp}_4(\mathbb{Z})\backslash\mathbb{H}_2} |F(Z)|^2 \phi(Z) (\det Y)^{k-3}\,\mathrm{d}X\mathrm{d}Y \longrightarrow \frac{1}{\mathrm{vol}(\mathrm{Sp}_4(\mathbb{Z})\backslash\mathbb{H}_2)} \int_{\mathrm{Sp}_4(\mathbb{Z})\backslash\mathbb{H}_2} \phi(Z)\,(\det Y)^{-3}\mathrm{d}X\mathrm{d}Y$$

as $k \longrightarrow \infty$.

[1] These two difficulties are also present in the case of half-integral weight forms and were overcome by the second-named author and Radziwiłł [LR20] under GRH; in Section 1.4 we discuss the relationship between their work and the present one.

A consequence of the classical holomorphic QUE theorem of Holowinsky and Soundararajan is that the set of zeros of a sequence of holomorphic Hecke cusp forms become equidistributed with respect to the hyperbolic measure as the weight tends to infinity [Rud05]. We are able to prove a similar result in the case $n = 2$ as a consequence of Theorem 1.2. For $F \in S_k(\mathrm{Sp}_4(\mathbb{Z}))$ we let Z_F denote the zero divisor of F, which we view as a current of integration (i.e., a distribution on the space of smooth compactly supported differential forms) of bidegree $(2,2)$ on $\mathrm{Sp}_4(\mathbb{Z})\backslash\mathbb{H}_2$.

We let

$$\omega := -\frac{i}{2\pi}\partial\overline{\partial}\log(\det Y)$$

be the Kähler differential form of bidegree $(1,1)$ on $\mathbb{H}_2$ associated to the Bergman metric on $\mathbb{H}_2$. Here ∂ and $\overline{\partial}$ are the Dolbeault operators and we write $Z \in \mathbb{H}_2$ as $Z = X + iY$. It is easy to see that ω descends to a differential form on $\mathrm{Sp}_4(\mathbb{Z})\backslash\mathbb{H}_2$. As an application of our theorem, we show that if F traverses a sequence of Saito–Kurokawa lifts with weights $k \longrightarrow \infty$, then the currents $\frac{1}{k}Z_F$ converge to ω weakly in the sense of measures.

Theorem 1.3. *Assume GRH. Fix a smooth compactly supported differential form η of bidegree $(2,2)$ on $\mathrm{Sp}_4(\mathbb{Z})\backslash\mathbb{H}_2$. Let $F \in S_k(\mathrm{Sp}_4(\mathbb{Z}))$ traverse a sequence of Hecke eigenforms that are Saito–Kurokawa lifts. Then*

$$\frac{1}{k}\int_{Z_F}\eta \longrightarrow \int_{\mathrm{Sp}_4(\mathbb{Z})\backslash\mathbb{H}_2}\omega\wedge\eta \tag{1.1}$$

as $k \longrightarrow \infty$.

Remark 1. *We remark that the only reason we assume GRH in Theorem 1.3 is that our mass equidistribution result Theorem 1.2 requires it. Note that in the proof of Theorem 1.3 we appeal to a sup norm result of Blomer, which is conditional under GRH, but a weaker bound for the sup norm that suffices can be shown unconditionally.*

1.3. **Overview of the proof.** We now sketch the main ideas behind the proof of Theorem 1.2. The starting point is to introduce a collection of incomplete Poincaré series on $\mathrm{Sp}_4(\mathbb{Z})\backslash\mathbb{H}_2$. One can attach such Poincaré series to any parabolic subgroup of $\mathrm{Sp}_4(\mathbb{R})$ but the best choice for our purposes is the Siegel parabolic (because its associated unipotent radical is abelian) which leads to the so-called Siegel-Poincaré series. More precisely, let Λ_2 be the set of 2 by 2 semi-integral symmetric matrices, i.e., $\Lambda_2 := \left\{\begin{pmatrix} m & r/2 \\ r/2 & n\end{pmatrix} : m,r,n \in \mathbb{Z}\right\}$. Given a symmetric semi-integral matrix $S \in \Lambda_2$ and a smooth compactly supported function h on $\mathrm{GL}_2(\mathbb{R})^+/\mathrm{SO}(2) \simeq \mathbb{H}\times\mathbb{R}^+$, we define an incomplete Siegel-Poincaré series $P_S^h \in C_c^\infty(\mathrm{Sp}_4(\mathbb{Z})\backslash\mathbb{H}_2)$ associated to this data. We show in Section 2.3 that the uniform span of the functions P_S^h obtained this way (as we vary S and h) equals the full space $C_c^\infty(\mathrm{Sp}_4(\mathbb{Z})\backslash\mathbb{H}_2)$. Therefore, proving Theorem 1.2 reduces to showing that for each fixed h and S as above, and a sequence of Saito–Kurokawa lifts $F \in S_k(\mathrm{Sp}_4(\mathbb{Z}))$ that are Hecke eigenforms,

$$\frac{1}{\|F\|_2^2}\int_{\mathrm{Sp}_4(\mathbb{Z})\backslash\mathbb{H}_2} P_S^h(Z)|F(Z)|^2(\det Y)^k\,\mathrm{d}\mu \longrightarrow \int_{\mathrm{Sp}_4(\mathbb{Z})\backslash\mathbb{H}_2} P_S^h(Z)\,\mathrm{d}\mu \quad \text{as } k\longrightarrow\infty,$$

where $\mathrm{d}\mu = (\det Y)^{-3}\mathrm{d}X\,\mathrm{d}Y$. We have two cases depending on whether S equals zero or not:

- (The off-diagonal case) For fixed h, S with $S \neq 0$, show that as $k \longrightarrow \infty$,

$$\frac{1}{\|F\|_2^2}\int_{\mathrm{Sp}_4(\mathbb{Z})\backslash\mathbb{H}_2} P_S^h(Z)|F(Z)|^2(\det Y)^k\,\mathrm{d}\mu \longrightarrow 0. \tag{1.2}$$

- (The diagonal case) For fixed h, show that as $k \longrightarrow \infty$,

$$\frac{1}{\|F\|_2^2} \int\limits_{\mathrm{Sp}_4(\mathbb{Z})\backslash\mathbb{H}_2} P_0^h(Z)|F(Z)|^2(\det Y)^k \,\mathrm{d}\mu \longrightarrow \int\limits_{\mathrm{Sp}_4(\mathbb{Z})\backslash\mathbb{H}_2} P_0^h(Z)\,\mathrm{d}\mu. \tag{1.3}$$

By unfolding the left-hand side of (1.2) or (1.3), we obtain the higher rank *shifted convolution* sum

$$\frac{1}{\|F\|_2^2} \sum_{T\in\Lambda_2} a(T)a(T+S)W_{h,S}(T),$$

where $a(T)$ are the Fourier coefficients of F and $W_{h,S}$ is a weight function which is roughly supported on those $T = \begin{pmatrix} m & r/2 \\ r/2 & n \end{pmatrix}$ for which $m, n, r \asymp_S k$. The crux of the proof of Theorem 1.2 lies in estimating these sums. To the best of our knowledge, such shifted convolution sums in higher rank where the Fourier coefficients are highly non-multiplicative have not been previously tackled successfully, even when the length of the sum does not depend on the spectral parameters.

We now briefly describe our treatment of the shifted convolution sum in the off-diagonal case. Due to the small range of summation over T there are no tools available that can obtain cancellation among the Fourier coefficients. However, we can exploit the fact that our form F is a Saito-Kurokawa lift and therefore its Fourier coefficients arise from those of a classical half-integral weight eigenform $\widetilde{f}$ of weight $k-\frac{1}{2}$ on $\Gamma_0(4)\backslash\mathbb{H}$. We forego obtaining cancellation in the shifted convolution problem and use Waldspurger's formula connecting squares of Fourier coefficients of half-integral weight forms with central values of L-functions to essentially reduce the problem to showing that

$$\frac{1}{k^3} \sum_{m,n,r\asymp k} \sqrt{L\left(\frac{1}{2}, f\otimes\chi_{r^2-4mn}\right) L\left(\frac{1}{2}, f\otimes\chi_{(r+\ell_2)^2-4(m+\ell_1)(n+\ell_3)}\right)} \longrightarrow 0 \tag{1.4}$$

as $k\longrightarrow\infty$, where f is an integral weight Hecke eigenform (of weight $2k-2$) associated to $\widetilde{f}$ by the Shimura correspondence, $S = \begin{pmatrix} \ell_1 & \ell_2/2 \\ \ell_2/2 & \ell_3 \end{pmatrix} \neq 0$ is fixed, and χ_d denotes the quadratic character associated to the discriminant d.

Proving the limit (1.4) unconditionally currently seems hopeless, as the techniques developed in [RlS15] to obtain bounds for fractional moments of central L-values require an asymptotic for a first moment that is well out of reach. We assume GRH and succeed in proving this bound under this assumption by using Soundararajan's method for bounding moments. This method involves several delicate and technical steps (including a rather involved character sum computation) which are performed in Section 4.1.

Next, we outline our treatment of the diagonal case. The left hand side of the sum (1.3) reduces to a sum in which the range of $\det T$ is too small to be evaluated asymptotically using a contour shifting argument. In previous works such as [Hol10], [HS10], [LR20] the analogous problem was resolved by introducing an auxiliary Eisenstein series to increase the length of the sum. This strategy seems hard to implement in our situation because of the complexity of the various types of Eisenstein series of higher rank and their Fourier coefficients.

Instead we introduce a completely new method for showing (1.3). The first step is to replace P_0^h by an incomplete Eisenstein series by doing an initial summation over $\mathrm{SL}_2(\mathbb{Z})$. By unfolding further and using Waldspurger's formula we are essentially reduced to estimating the sum over negative discriminants

$$\sum h(d)L\left(\tfrac{1}{2}, f\otimes\chi_d\right) G(d,g;\kappa), \tag{1.5}$$

where $\kappa \in C_c^\infty(\mathbb{R}^+)$, $g \in L^2(\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H})$ are fixed, $h(d)$ is the cardinality of the class group Cl_d, and $G(d; g, \kappa)$ is a weight function that (up to some simple factors depending only on k) is equal to

$$\frac{|d|^{k-\frac{3}{2}}}{h(d)} \sum_{T \in \mathrm{Cl}_d} \int\limits_{Y \in M_2^{\mathrm{sym}}(\mathbb{R})^+} g(Y^{1/2} \cdot i)(\det Y)^{k-3} \kappa(\sqrt{\det Y}) e^{-4\pi \mathrm{Tr}(TY)} \, \mathrm{d}Y$$

(see Section 1.7 for clarification on the notation).

Our main term arises from the case where $g = 1$, as $G(d, 1; \kappa)$ acts as a smooth weight function that localizes the sum to $|d| \asymp k^2$. Since the ratio of the logarithms of the analytic conductor of $L(\frac{1}{2}, f \otimes \chi_d)$ and the length of the sum is $\log(|d|^2 k^2)/\log k^2 \sim 3$, the moment estimate we require does not yield a subconvex estimate for the central L-values and is amenable to the methods developed in [Sou00, SY10]. To implement this, we prove a twisted first moment asymptotic for L-functions on GL_2 assuming GLH[2] (see Section 5.1) and then combine this result with delicate computations (Section 5.2) involving the residue of the Rankin–Selberg convolution of the Koecher–Maass series. This enables us to obtain the required limit for (1.5) in the case of $g = 1$.

To estimate (1.5) in the case where g is orthogonal to the constant function, we develop a new method that morally boils down to appealing to the famous equidistribution of Heegner points as $|d| \longrightarrow \infty$. We use Waldspurger's formula on toric integrals and the subconvexity bound for $L(\frac{1}{2}, g \otimes \chi_d)$ to show that the sum (over the class group elements) that occurs in the definition of $G(d, g; \kappa)$ has a nontrivial cancellation that saves a *power* of $|d|$. More precisely, we prove that for all Hecke eigenforms $g \in L^2(\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H})$ orthogonal to the constant function, we have

$$G(d, g; \kappa) \ll_{g,\varepsilon} d^{-\frac{1}{12}+\varepsilon} G(d, 1; |\kappa|). \tag{1.6}$$

Therefore, the size of $|G(d, g; \kappa)|$ is quite small in comparison to $G(d, 1; |\kappa|)$, which bounds (1.5) and completes the proof of (1.3) as a consequence of the previously-proved subcase where $g = 1$.

1.4. **Comparison with other work.** Not much was previously known in the direction of Conjecture 1.1 in the higher rank setting $n > 1$. Liu [Liu17] established the limit $D_F(\phi) \longrightarrow 0$ when the test function ϕ is a degenerate Klingen Eisenstein series *and* F traverses a sequence of Ikeda lifts. More recently, Katsurada and Kim [KK22] proved a similar result when the test function ϕ is a degenerate Siegel Eisenstein series and F again traverse a sequence of Ikeda lifts, under the additional assumptions that $n \geq 4$ and a certain Dirichlet series is meromorphic. The techniques used in those papers are very different from the ones used in this work.

Arguably the work that is closest in spirit to this paper is that of the second named author and Radziwiłł [LR20] who proved the mass equidistribution for the family of classical half-integral weight Hecke eigenforms on $\Gamma_0(4)\backslash\mathbb{H}$ (both in the weight aspect and the eigenvalue aspect). For the proof, as in the present paper, they consider a family of incomplete Poincaré series and reduce to a shifted convolution sum. However, there are key differences between [LR20] and the present work.

The treatment of the diagonal case in [LR20] is completely different from that in the present work. As mentioned earlier, the second named author and Radziwiłł adapted a technique of Holowinsky and introduced an auxiliary Eisenstein series to increase the length of the sum. This idea seems hard to implement in our case due to the complexity of the various types of Eisenstein series on rank 2 symplectic groups and the lack of precise information about their Fourier coefficients. Instead, we introduce a completely different method to deal with the diagonal case. We build upon an adelic version of the equidistribution of Heegner points (Waldspurger's period formula for toric integrals and the subconvex bounds for twisted L-functions) to reduce the diagonal case to a special subcase

[2]Throughout the article, GLH refers to the Generalized Lindelöf Hypothesis.

that is proved ultimately by reducing to a twisted first moment asymptotic for L-functions attached to twists of holomorphic newforms.

In the off-diagonal case, the second named author and Radziwiłł essentially reduce the problem to showing that

$$\frac{1}{k}\sum_{d\asymp k}\sqrt{L\left(\frac{1}{2}, f\otimes\chi_d\right)L\left(\frac{1}{2}, f\otimes\chi_{d+\ell}\right)}\longrightarrow 0 \tag{1.7}$$

as $k\longrightarrow\infty$, where f is an integral weight Hecke eigenform (of weight $2k$), $\ell\neq 0$ is fixed, and χ_d denotes the quadratic character associated to the discriminant d. The corresponding problem in our case is given by (1.4). Note that in our case, there is a shift in each of the matrix entries, rather than just a shift of the discriminant and this leads to a significantly increased complexity in implementing Soundararajan's method to prove (1.4) that goes beyond the intricate estimates used to establish (1.7). An indication of the difference in the difficulties involved can be seen by comparing [LR20, Prop 3.1] with the proof of Proposition 4.3 of this paper.

1.5. **Plan for the paper.** Our paper is organized as follows. In Section 2, we develop the theory of Poincaré series associated to the Siegel parabolic of Sp_4 and reduce Theorem 1.2 to the case where the test function ϕ is a Poincaré series associated to factorizable data. In Section 3 we reduce further to proving two assertions involving estimates on higher rank shifted convolution sums. Section 4 is devoted to the proof of the first of these two assertions, which corresponds to the off-diagonal case. Section 5 is devoted to the proof of the second assertion corresponding to the diagonal case. Finally, in Section 6, we use Theorem 1.2 to deduce Theorem 1.3.

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1.7. **Notation.**

1.7.1. *General.* We use the notation $A\ll_{x,y,z} B$ to signify that there exists a positive constant C, depending at most upon x, y, z, so that $|A|\leq C|B|$. The symbol ε will denote a small positive quantity. We write $A(x)=O_y(B(x))$ if there exists a positive real number M (depending on y) and a real number x_0 such that $|A(x)|\leq M|B(x)|$ for all $x\geq x_0$.

For a smooth orbifold X, we let $C_b(X)$ denote the space of bounded continuous functions $X\longrightarrow\mathbb{C}$, $C_c(X)$ denote the space of compactly supported continuous functions $X\longrightarrow\mathbb{C}$, and $C_c^\infty(X)$ denote the space of compactly supported smooth functions $X\longrightarrow\mathbb{C}$. We say a function $g:\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}\longrightarrow\mathbb{C}$ is slowly growing if $g(x+iy)\ll_N y^N+y^{-N}$.

We let $\mathcal{D}$ denotes the set of negative fundamental discriminants. Given an integer n and prime p we write $p^a||n$ if $p^a|n$ and $p^{a+1}\nmid n$. Also, we define $\Omega(n)=\sum_{p^a||n}a$. Additionally, for $a,b\in\mathbb{Z}$ and $c\in\mathbb{N}$ we write $a\equiv b\,(c)$ which means $a\equiv b\pmod{c}$.

We let $\mathbb{R}$ denote the reals and let $\mathbb{R}^+$ denote the positive reals. For $\kappa\in C_c^\infty(\mathbb{R}^+)$, we define the Mellin transform (note that our definition is nonstandard)

$$\widetilde{\kappa}(s):=\int_0^\infty\kappa(\lambda)\lambda^{-s-1}\mathrm{d}\lambda$$

so that by the inversion formula we have for all $\sigma \geq 2$

$$\kappa(y) = \frac{1}{2\pi i} \int\limits_{(\sigma)} \widetilde{\kappa}(s) y^s \mathrm{d}s.$$

Similarly for sufficiently nice $h \in C^\infty(\mathbb{R})$ we define the Fourier transform

$$\widehat{h}(\xi) := \int\limits_{-\infty}^{\infty} h(x) e(-x\xi)\, \mathrm{d}x,$$

which satisfies the Fourier inversion formula

$$h(x) = \int\limits_{-\infty}^{\infty} \widehat{h}(\xi) e(x\xi)\, \mathrm{d}\xi.$$

Throughout the article we write $e(x) := e^{2\pi i x}$.

1.7.2. *Matrix groups.* For a positive integer n and a commutative ring R, we let $M_n(R)$ denote the ring of $n \times n$ matrices over R, and $\mathrm{GL}_n(R)$ the multiplicative subgroup of invertible matrices in $M_n(R)$. We let $M_n^{\mathrm{Sym}}(R)$ be the additive subgroup of symmetric matrices in $M_n(R)$. Let I_n denote the n by n identity matrix. Given $A \in M_n^{\mathrm{Sym}}(\mathbb{R})$ and $c \in R$ we write $A > c$ (resp. $A \geq c$) if $A - cI_n$ is positive definite (resp., positive semidefinite). Denote by J_n the $2n$ by $2n$ matrix given by

$$J_n := \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}.$$

Define

$$\begin{aligned} \mathrm{GSp}_{2n}(R) &:= \left\{ g \in \mathrm{GL}_{2n}(R) \,:\, {}^t g J g = \mu_n(g) J_n,\ \mu_n(g) \in R^\times \right\} \\ \mathrm{Sp}_{2n}(R) &:= \{ g \in \mathrm{GSp}_{2n}(R) \,:\, \mu_n(g) = 1 \} \end{aligned}$$

We also set

$$\begin{aligned} \mathrm{GL}_n(\mathbb{R})^+ &:= \{ g \in \mathrm{GL}_n(\mathbb{R}) \,:\, \det(g) > 0 \} \\ \mathrm{GSp}_{2n}(\mathbb{R})^+ &:= \{ g \in \mathrm{GSp}_{2n}(\mathbb{R}) \,:\, \mu_n(g) > 0 \} \\ M_n^{\mathrm{Sym}}(\mathbb{R})^+ &:= \left\{ g \in M_n^{\mathrm{Sym}}(\mathbb{R}) \,:\, g \text{ is positive definite} \right\} \end{aligned}$$

We denote

$$G := \mathrm{Sp}_4(\mathbb{R}), \quad \Gamma := \mathrm{Sp}_4(\mathbb{Z}).$$

For $X \in M_2^{\mathrm{Sym}}(\mathbb{R})$, let $n(X) := \begin{pmatrix} I_2 & X \\ 0 & I_2 \end{pmatrix} \in G$. For $A \in \mathrm{GL}_2(\mathbb{R})$, let $m(A) := \begin{pmatrix} A & 0 \\ 0 & {}^t A^{-1} \end{pmatrix} \in G$. Denote

$$\begin{aligned} N(\mathbb{R}) &:= \{ n(X) : X \in M_2^{\mathrm{Sym}}(\mathbb{R}) \}, \\ M(\mathbb{R})^+ &:= \{ m(A) : A \in \mathrm{GL}_2(\mathbb{R})^+ \}. \end{aligned}$$

Let

$$\begin{aligned} \Gamma_\infty &:= N(\mathbb{Z}) = \{ n(X) : X \in M_2^{\mathrm{Sym}}(\mathbb{Z}) \}, \\ P(\mathbb{Z}) &:= \{ n(X) m(A) : X \in M_2^{\mathrm{Sym}}(\mathbb{Z}), A \in \mathrm{SL}_2(\mathbb{Z}) \}, \end{aligned}$$

and note that $\Gamma_\infty \subset P(\mathbb{Z}) \subset \Gamma$. Let K_∞ be the standard maximal compact subgroup of G consisting of all elements of the form $\begin{pmatrix} A & B \\ -B & A \end{pmatrix}$; it can be checked that K_∞ is the subgroup of G fixing the

point iI_2. We have a natural identification $G/K_\infty \simeq \mathbb{H}_2$ sending g to $g\langle iI_2\rangle$, where $\mathbb{H}_2$ and $g\langle Z\rangle$ are as defined in the next subsection. We also have the Iwasawa decomposition

$$G = N(\mathbb{R})M(\mathbb{R})^+K_\infty.$$

1.7.3. *Modular forms.* Let

$$\mathbb{H}_n := \{Z \in M_n(\mathbb{C}) \,:\, Z = {}^tZ, \,\mathrm{Im}(Z) \,\text{is positive definite}\}.$$

We will often write elements $Z \in \mathbb{H}_n$ as $Z = X + iY$ for $X \in M_n^{\mathrm{Sym}}(\mathbb{R})$, $Y \in M_n^{\mathrm{Sym}}(\mathbb{R})^+$. For $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{GSp}_{2n}(\mathbb{R})^+$ and $Z \in \mathbb{H}_n$, define

$$g\langle Z\rangle := (AZ + B)(CZ + D)^{-1}, \quad J(g, Z) := CZ + D.$$

We will sometimes shorten $g\langle Z\rangle$ to $g \cdot Z$ or gZ when the meaning is clear from the context. The space $S_k(\Gamma)$ consists of holomorphic functions $F : \mathbb{H}_2 \longrightarrow \mathbb{C}$ which satisfy the relation

$$F(\gamma\langle Z\rangle) = \det(J(\gamma, Z))^k F(Z)$$

for $\gamma \in \Gamma$, $Z \in \mathbb{H}_2$, and vanish at all the cusps.

Let Λ_2 denote the set of 2 by 2 semi-integral symmetric matrices, i.e.,

$$\Lambda_2 = \left\{ \begin{pmatrix} m & r/2 \\ r/2 & n \end{pmatrix} : m, r, n \in \mathbb{Z} \right\},$$

and Λ_2^+ be the subset of positive definite matrices in Λ_2. For $S = \begin{pmatrix} m & r/2 \\ r/2 & n \end{pmatrix} \in \Lambda_2$, we define its discriminant $\mathrm{disc}(S) := -4 \det S = r^2 - 4mn$ and its content $\mathrm{cont}(S) := \gcd(m, r, n)$. The group $\mathrm{SL}_2(\mathbb{Z})$ acts on Λ_2 on the right via $S \mapsto {}^tASA$ and this action preserves the discriminant and the content. Let $\mathcal{D}$ denote the set of negative fundamental discriminants.

1.7.4. *Adeles and L-functions.* We let $\mathbb{A} = \otimes_v' \mathbb{Q}_v$ denote the ring of adeles over $\mathbb{Q}$ and $\mathbb{A}_f = \otimes_p' \mathbb{Q}_p$ the subring of finite adeles. Given a reductive group G such that the centre of $G(\mathbb{A})$ is isomorphic to $\mathbb{A}^\times$ we let $Z(\mathbb{A})$ denote the centre of $G(\mathbb{A})$ (the group G involved should be clear from context). For an automorphic representation π of $\mathrm{GL}_n(\mathbb{A})$ we let $L(s, \pi) = \prod_{p<\infty} L(s, \pi_p)$ denote the finite part of the L-function (i.e., without the archimedean factors), and normalized so that it satisfies a functional equation under $s \mapsto 1 - s$. For a positive integer M we denote $L^M(s, \pi) = \prod_{p \nmid M} L(s, \pi_p)$.

2. Siegel-Poincaré series

In this Section we reduce Theorem 1.2 to the case where the test function ϕ is a Poincaré series associated to the Siegel parabolic subgroup. The key result of this section which will be used in the rest of the paper is Corollary 2.6.

2.1. **Reduction to smooth compactly supported functions.** For $F \in S_k(\Gamma)$ and a function $\phi \in C_b(\Gamma\backslash\mathbb{H}_2)$, define

$$\mu_F(\phi) = \int\limits_{\Gamma\backslash\mathbb{H}_2} |F(Z)|^2 \phi(Z) (\det Y)^k \, \mathrm{d}\mu,$$

where $d\mu = (\det Y)^{-3} dX \, dY$ denotes the standard measure on $\Gamma\backslash\mathbb{H}_2$.

Let D_F be the linear functional on $C_b(\Gamma\backslash\mathbb{H}_2)$ defined by

$$D_F(\phi) = \frac{\mu_F(\phi)}{\mu_F(1)} - \frac{1}{\mathrm{vol}(\Gamma\backslash\mathbb{H}_2)} \int\limits_{\Gamma\backslash\mathbb{H}_2} \phi(Z) \, \mathrm{d}\mu.$$

Proposition 2.1. *Let F_i be a sequence consisting of elements in $S_k(\Gamma)$. Suppose that for each ϕ in $C_c^\infty(\Gamma\backslash\mathbb{H}_2)$ we have $D_{F_i}(\phi) \longrightarrow 0$. Then for each ϕ in $C_b(\Gamma\backslash\mathbb{H}_2)$ we have $D_{F_i}(\phi) \longrightarrow 0$.*

Proof. Since the space $C_c^\infty(\Gamma\backslash\mathbb{H}_2)$ is dense (in the uniform topology) in the space $C_c(\Gamma\backslash\mathbb{H}_2)$, it follows immediately that

$$\text{for each } \phi \in C_c(\Gamma\backslash\mathbb{H}_2) \text{ we have } D_{F_i}(\phi) \longrightarrow 0. \tag{2.1}$$

Now fix a function ϕ in $C_b(\Gamma\backslash\mathbb{H}_2)$. We need to show that $D_{F_i}(\phi) \longrightarrow 0$. Let $\varepsilon > 0$ be arbitrary. Let $D \subset \mathbb{H}_2$ be the standard fundamental domain for $\Gamma\backslash\mathbb{H}_2$ as described in [Kli90, page 30]. For $T > 0$, let C_T be the compact subset of $\Gamma\backslash\mathbb{H}_2$ given by the image of the set $\{X + iY \in D,\ Y \leq TI_2\}$ in $\Gamma\backslash\mathbb{H}_2$ and let B_T be the complement of C_T in $\Gamma\backslash\mathbb{H}_2$. Choose $T = T(\varepsilon)$ large enough that $\mu(B_T)/\mathrm{vol}(\Gamma\backslash\mathbb{H}_2) < \varepsilon$. It is clear that we can write $\phi = \phi_1 + \phi_2$, where $\phi_1 \in C_c(\Gamma\backslash\mathbb{H}_2)$ and ϕ_2 is supported on B_T. By (2.1) $|D_{F_i}(\phi_1)| < \varepsilon$ eventually.[3] Choose a smooth $[0,1]$-valued function h supported on C_T that satisfies $\int_{\Gamma\backslash\mathbb{H}_2} h(Z)d\mu > \mathrm{vol}(\Gamma\backslash\mathbb{H}_2)(1-2\varepsilon)$. Then (2.1) implies that the positive real number $\mu_{F_i}(h)/\mu_{F_i}(1)$ eventually exceeds $1 - 3\varepsilon$. By the nonnegativity of $\mu_{F_i}(\chi)$ for all nonnegative valued functions χ, we deduce that $\mu_{F_i}(\chi_{B_T})/\mu_{F_i}(1) < 3\varepsilon$ eventually, where χ_{B_T} denotes the characteristic function of B_T. Let R be the supremum of $|\phi|$. Then $|\mu_{F_i}(\phi_2)/\mu_{F_i}(1)| \leq R\mu_{F_i}(\chi_{B_T})/\mu_{F_i}(1) \leq 3R\varepsilon$ eventually and $\mathrm{vol}(\Gamma\backslash\mathbb{H}_2)^{-1}|\int_{\Gamma\backslash\mathbb{H}_2}\phi(Z)d\mu| \leq R\varepsilon$, so that $|D_{F_i}(\phi_2)| \leq 4R\varepsilon$ eventually. Thus $|D_{F_i}(\phi)| < (1+4R)\varepsilon$ eventually. This completes the proof. □

2.2. **Definition of Poincaré series.** Let

$$\mathcal{M}_2 := \mathrm{GL}_2(\mathbb{R})^+/\mathrm{SO}(2).$$

Recall that $M_2^{\mathrm{Sym}}(\mathbb{R})^+$ is the set of symmetric positive definite 2×2 matrices over $\mathbb{R}$, which we may view as a smooth manifold. We have a diffeomorphism

$$\iota : \mathcal{M}_2 \xrightarrow{\simeq} M_2^{\mathrm{Sym}}(\mathbb{R})^+, \quad \iota(A) := A\ {}^tA.$$

Note that for each $Y \in M_2^{\mathrm{Sym}}(\mathbb{R})^+$, $\iota^{-1}(Y)$ equals the class of $Y^{1/2}$ in $\mathcal{M}_2$. The basic input for our Poincaré series on $\Gamma\backslash\mathbb{H}_2$ is a pair (h, S) where $h \in C_c^\infty(\mathcal{M}_2)$ and $S \in \Lambda_2$.

For $h \in C_c^\infty(\mathcal{M}_2)$ and $S \in \Lambda_2$, define the function h_S on G via the Iwasawa decomposition as follows:

$$h_S(n(X)m(A)k) := e(\mathrm{Tr}(SX))h(A), \quad A \in \mathrm{GL}_2(\mathbb{R})^+,\ X \in M_2^{\mathrm{Sym}}(\mathbb{R}),\ k \in K_\infty.$$

It is easy to check this is well-defined. Since h_S is right K_∞-invariant, it defines a function on $G/K_\infty \simeq \mathbb{H}_2$ which we also denote as h_S. Concretely, for $Z = X + iY \in \mathbb{H}_2$, we have

$$h_S(Z) = e(\mathrm{Tr}(SX))h(\iota^{-1}(Y)).$$

We define the Poincaré series $P_S^h(g)$ on G via

$$P_S^h(g) := \sum_{\gamma \in \Gamma_\infty\backslash\Gamma} h_S(\gamma g). \tag{2.2}$$

The above sum is in fact finite due to the compact support of h, as shown in Lemma 2.2 below. It is clear that $P_S^h(g)$ is left Γ-invariant and right K_∞-invariant, and hence defines a function on $G/K_\infty \simeq \mathbb{H}_2$ and on $\Gamma\backslash G/K_\infty \simeq \Gamma\backslash\mathbb{H}_2$. By abuse of notation, we will also denote these functions as P_S^h.

[3]Here and in what follows, "eventually" means "provided that i large enough".

Note that $\mathrm{Im}((n(X)m(A)k)\cdot iI_2)=\iota(A)$ for all $A\in \mathrm{GL}_2(\mathbb{R})^+$, $X\in M_2^{\mathrm{Sym}}(\mathbb{R})$, and $k\in K_\infty$. From this and the definitions, it follows that for $Z\in\mathbb{H}_2$ we have the formula

$$P_S^h(Z)=\sum_{\gamma\in\Gamma_\infty\backslash\Gamma} h_S(\gamma Z)=\sum_{\gamma\in\Gamma_\infty\backslash\Gamma} e(\mathrm{Tr}(S\mathrm{Re}(\gamma Z)))h(\iota^{-1}(\mathrm{Im}(\gamma Z))). \tag{2.3}$$

Lemma 2.2. *For each $h\in C_c^\infty(\mathcal{M}_2)$ and $S\in\Lambda_2$, we have $P_S^h(Z)\in C_c^\infty(\Gamma\backslash\mathbb{H}_2)$. Furthermore, if h is supported on some compact set C, then there exists a compact subset D_C of $\Gamma\backslash\mathbb{H}_2$ and a positive integer N_C, with both D_C and N_C depending only on C, such that P_S^h is supported on D_C, and the sum* (2.3) *defining P_S^h has at most N_C nonzero terms.*

Proof. Since $\iota(C)$ is compact, there exist positive constants a_C, b_C such that

$$0<b_C\le Y\le a_C \tag{2.4}$$

for all $Y\in\iota(C)$. Let $D\subset\mathbb{H}_2$ be the standard fundamental domain for $\Gamma\backslash\mathbb{H}_2$ as described in [Kli90, page 30], and recall that $\mathrm{Im}(Z)\gg 1$ for all $Z\in D$. We will consider P_S^h as a function on $\mathbb{H}_2$ and consider the support of $P_S^h|_D$. So, suppose $P_S^h(Z_0)\neq 0$ for $Z_0=X_0+iY_0\in D$. The expression (2.3) shows that there exists some $\gamma\in\Gamma$ such that $\mathrm{Im}(\gamma Z_0)\in\iota(C)$. Set $\gamma Z_0=Z=X+iY$ so that $Z_0=\gamma^{-1}Z$. Put $\gamma^{-1}=\begin{pmatrix}P & Q\\ R & S\end{pmatrix}$. Now the formula (see [Kli90, page 8])

$$Y_0^{-1}=(RX+S)Y^{-1}\,{}^t(RX+S)+RY\,{}^tR$$

shows that $Y_0^{-1}\ge SY^{-1}{}^tS+RY\,{}^tR$. Since R,S are both integral and not both equal to 0, it follows from (2.4) that $Y_0^{-1}\ge\min(a_C^{-1},b_C)$, and hence that $Y_0\le\max(a_C^{-1},b_C)$. Thus we have shown Z_0 is contained in a compact set $C'\subset D$ depending on C. Now take D_C to the image of C' in $\Gamma\backslash\mathbb{H}_2$; then D_C is compact and $P_S^h(Z_0)=0$ for $Z_0\notin D_C$.

Finally, we show that the sum (2.3) defining P_S^h has at most N_C nonzero terms. Let $R_C\subset\mathbb{H}_2$ be the compact set consisting of all $Z=X+iY$ such that $Y\in\iota(C)$, $-\frac{1}{2}\le X_{i,j}\le\frac{1}{2}$. Because the action of Γ on $\mathbb{H}$ is properly discontinuous and because C' and R_C are compact, it follows that the cardinality of the set

$$S_C:=\{\gamma\in\Gamma:\gamma C'\cap R_C\neq\emptyset\}$$

is finite. We let N_C denote the cardinality of S_C; the proof follows from the observation that any $\gamma\in\Gamma_\infty\backslash\Gamma$ that contributes nontrivially to (2.3) must have a representative in S_C. □

2.3. **Uniform approximation by Poincaré series.**

Proposition 2.3. *The set of finite linear combinations of Poincaré series P_S^h with $h\in C_c^\infty(\mathcal{M}_2)$ and $S\in\Lambda_2$ is dense in the space $C_c^\infty(\Gamma\backslash\mathbb{H}_2)$ equipped with the uniform topology, i.e., for $\phi\in C_c^\infty(\Gamma\backslash\mathbb{H}_2)$ and $\varepsilon>0$, there exists a function*

$$P_0=\sum_{i=1}^r a_iP_{S_i}^{h_i}$$

with $h_i\in C_c^\infty(\mathcal{M}_2)$, $S_i\in\Lambda_2$ and $a_i\in\mathbb{R}$ that satisfies

$$|\phi(Z)-P_0(Z)|<\varepsilon,\ \textit{for all } Z\in\Gamma\backslash\mathbb{H}_2. \tag{2.5}$$

Proof. Let B_ϕ be a compact subset of $\mathbb{H}_2$ whose image in $\Gamma\backslash\mathbb{H}_2$ contains the support of ϕ.

Let $\bar\Gamma:=\{\pm1\}\backslash\Gamma$. For each $Z\in B_\phi$ that is *not* fixed by any nontrivial element of $\bar\Gamma$, pick a fundamental domain $D_Z\subset\mathbb{H}_2$ for the action of Γ, and an open neighbourhood B_Z of Z satisfying $Z\in B_Z\subset D_Z\subset\mathbb{H}_2$. Let C_Z be the image of B_Z in $\Gamma\backslash\mathbb{H}_2$ so that the natural map $B_Z\xrightarrow{\simeq}C_Z$ is a diffeomorphism.

For each point $Z \in B_\phi$ that *is* fixed[4] by a nontrivial element of $\bar{\Gamma}$, let $\bar{\Gamma}_Z$ be the stabilizer of Z in $\bar{\Gamma}$ and pick a fundamental domain $D_Z \subset \mathbb{H}_2$ for the action of Γ that contains Z and pick also an open neighbourhood $B_Z \ni Z$ intersecting D_Z and having the property that $B_Z = \cup_{\gamma \in \bar{\Gamma}_Z} \gamma(B_Z \cap D_Z)$. We let C_Z be the image of B_Z in $\Gamma\backslash\mathbb{H}_2$ and the note that the natural map $B_Z \longrightarrow C_Z$ induces a diffeomorphism $\bar{\Gamma}_Z\backslash B_Z \xrightarrow{\simeq} C_Z$.

Now, by the compactness of B_ϕ, there exist a finite set of points $Z_i,\ 1 \leq i \leq r$, with $Z_i \in B_\phi$ and $B_\phi \subset \bigcup_{i=1}^r B_{Z_i}$. For brevity, write $B_i := B_{Z_i},\ C_i := C_{Z_i}, D_i := D_{Z_i}$. By choosing a partition of unity subordinate to the open cover C_i, we may write $\phi = \sum_{i=1}^r \phi_i$ with the function $\phi_i \in C_c^\infty(\Gamma\backslash\mathbb{H}_2)$ supported on C_i. It is sufficient to show that each ϕ_i can be uniformly approximated by a finite linear combination of Poincaré series. *So for the rest of the proof, we can and will assume that*

$$\phi = \phi_j \text{ for some fixed } 1 \leq j \leq r.$$

Case I: Z_j *is not fixed by any nontrivial element of* $\bar{\Gamma}$. Let $\phi_0 \in C_c^\infty(\mathbb{H}_2)$ be the function that coincides with ϕ_j on D_j and is equal to 0 outside D_j (the smoothness of ϕ_0 uses the fact that the support C_j of ϕ_j is diffeomorphic to the open set B_j contained in the *interior* of D_j). We define

$$\tilde{\phi_0}(Z) := \sum_{\gamma \in \Gamma_\infty} \phi_0(\gamma Z).$$

Then $\tilde{\phi_0}$ is a smooth Γ_∞-invariant function on $\mathbb{H}_2$, and there exists a compact set $C_0 \subset M_2^{\mathrm{Sym}}(\mathbb{R})^+$ with the property that $\tilde{\phi_0}(X + iY) = 0$ for all $Y \notin C_0$ (we may take $C_0 = \overline{B_j}$).

From the fact that $N(\mathbb{R})$ is abelian and the fact that $\tilde{\phi_0}$ is a smooth function determined by its values on the compact set $\{X + iY : |X_{k,l}| \leq \frac{1}{2}, Y \in C_0\}$, we obtain a Fourier expansion converging absolutely and uniformly on $\mathbb{H}_2$:

$$\tilde{\phi_0}(Z) = \sum_{S \in \Lambda_2} a_0(S, Y) e(\mathrm{Tr}(SX)), \tag{2.6}$$

where for each $S \in \Lambda_2$, the function $Y \mapsto a_0(S, Y)$ is given by

$$a_0(S, Y) := \int\limits_{M_2^{\mathrm{Sym}}(\mathbb{Z})\backslash M_2^{\mathrm{Sym}}(\mathbb{R})} \tilde{\phi_0}(X + iY) e(-\mathrm{Tr}(SX)) dX.$$

It is clear that $Y \mapsto a_0(S, Y)$ is smooth and supported on C_0; moreover (using partial integration) we see that it is rapidly decaying in S. Precisely, given any $\mu > 0$, we have

$$\sup_{Y \in C_0} |a_0(S, Y)| \ll_{\phi_0, \mu} (1 + |S|)^{-\mu}, \tag{2.7}$$

where for $S = \begin{pmatrix} m & r/2 \\ r/2 & n \end{pmatrix}$ we denote $|S| := |m| + |r| + |n|$.

On the other hand, using (2.6), the definition (2.3) of Poincaré series, and the fact that any element in $\mathbb{H}_2$ is contained in $\gamma \cdot D_j$ for exactly 2 elements $\gamma \in \Gamma$, we obtain the absolutely convergent expression

$$\phi_j(Z) = \frac{1}{2} \sum_{S \in \Lambda_2} P_S^{\phi_S}(Z), \tag{2.8}$$

where the function ϕ_S on $\mathcal{M}_2$ is defined via

$$\phi_S(A) := a_0(S, \iota(A)) = a_0(S, A\ {}^tA).$$

[4]There are only finitely many such points.

A priori, the expression (2.8) converges pointwise for each Z, but we need to show that the convergence is uniform. For this, first observe that the functions ϕ_S are all supported on the compact set $\phi^{-1}(C_0)$. Then Lemma 2.2 implies that for each $S \in \Lambda_2$,

$$|P_S^{\phi_S}(Z)| \leq N_{C_0} \sup_{A \in \phi^{-1}(C_0)} |\phi_S(A)| = N_{C_0} \sup_{Y \in C_0} |a_0(S,Y)|,$$

so that

$$\frac{1}{2} \sum_{\substack{S \in \Lambda_2 \\ |S|>M}} |P_S^{\phi_S}(Z)| \leq \frac{1}{2} N_{C_0} \sum_{\substack{S \in \Lambda_2 \\ |S|>M}} \sup_{Y \in C_0} |a_0(S,Y)|. \tag{2.9}$$

Let $\varepsilon > 0$. Using (2.7), we pick M_ε such that

$$\frac{1}{2} N_{C_0} \sum_{\substack{S \in \Lambda_2 \\ |S|>M_\varepsilon}} \sup_{Y \in C_0} |a_0(S,Y)| < \varepsilon. \tag{2.10}$$

It follows from (2.8), (2.9) and (2.10) that

$$\left| \phi_j(Z) - \frac{1}{2} \sum_{\substack{S \in \Lambda_2 \\ |S| \leq M_\varepsilon}} P_S^{\phi_S}(Z) \right| < \varepsilon.$$

This completes the proof of (2.5) in this case.

Case II: *Z_j is fixed by any nontrivial element of $\bar{\Gamma}$.* The proof is essentially the same, so we indicate the main changes below. In this case we let $\phi_0 \in C_c^\infty(\mathbb{H}_2)$ be the function that coincides with ϕ_j on $\cup_{\gamma \in \bar{\Gamma}_{Z_j}} \gamma D_j$ and is equal to 0 outside it (the smoothness of ϕ_0 uses the fact that ϕ_0 is supported on the open set $B_j = \cup_{\gamma \in \bar{\Gamma}_{Z_j}} \gamma(B_j \cap D_j)$ contained in the *interior* of $\cup_{\gamma \in \bar{\Gamma}_{Z_j}} \gamma D_j$). We again define

$$\tilde{\phi_0}(Z) = \sum_{\gamma \in \Gamma_\infty} \phi_0(\gamma Z).$$

Then $\tilde{\phi_0}$ is a smooth Γ_∞-invariant function on $\mathbb{H}_2$, and there exists a compact set $C_0 \subset M_2^{\mathrm{Sym}}(\mathbb{R})^+$ with the property that $\tilde{\phi_0}(X+iY) = 0$ for all $Y \notin C_0$. As in the previous case we obtain a Fourier expansion converging absolutely and uniformly on $\mathbb{H}_2$:

$$\tilde{\phi_0}(Z) = \sum_{S \in \Lambda_2} a_0(S,Y) e(\mathrm{Tr}(SX)), \tag{2.11}$$

and the absolutely convergent expression

$$\phi_j(Z) = \frac{1}{2|\bar{\Gamma}_{Z_j}|} \sum_{S \in \Lambda_2} P_S^{\phi_S}(Z). \tag{2.12}$$

The rest of the proof is identical to the previous case. □

As an immediate corollary, we see that it is sufficient to prove Theorem 1.2 for test functions ϕ that are equal to P_S^h for some h, S.

Corollary 2.4. *Let F_i be a sequence consisting of elements in $S_k(\Gamma)$. Suppose that for each $h \in C_c^\infty(\mathcal{M}_2)$ and $S \in \Lambda_2$, we have $D_{F_i}(P_S^h) \longrightarrow 0$ as $i \longrightarrow \infty$. Then for each ϕ in $C_b(\Gamma \backslash \mathbb{H}_2)$ we have $D_{F_i}(\phi) \longrightarrow 0$ as $i \longrightarrow \infty$.*

Proof. This is immediate from Propositions 2.1 and 2.3. □

2.4. **Reduction of the proof to test functions coming from Poincaré series.** We now refine Corollary 2.4 by restricting h to certain *factorizable* functions. To make this precise, we note below two convenient ways to parameterize functions $h \in C_c^\infty(\mathcal{M}_2)$. The first parameterization relies on the isomorphism $\mathcal{M}_2 \simeq \mathbb{H} \times \mathbb{R}^\times$ given by $A \mapsto (A \cdot i, \det(A))$.

- Given a function ψ on $\mathbb{H} \times \mathbb{R}^+$, we obtain a function $h = h^\psi$ on $\mathcal{M}_2$ via
$$h(A) = \psi(A \cdot i, \det(A)).$$
Every function on $\mathcal{M}_2$ arises this way and $h^\psi \in C_c^\infty(\mathcal{M}_2)$ iff $\psi \in C_c^\infty(\mathbb{H} \times \mathbb{R}^+)$.
- Given a function ϕ on $\mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}$, we obtain a function $h = h^\phi$ on $\mathcal{M}_2$ via
$$h\left(\begin{pmatrix} 1 & u \\ & 1 \end{pmatrix}\begin{pmatrix} \sqrt{t_1} & \\ & \sqrt{t_2} \end{pmatrix}k\right) = \phi(t_1, t_2, u)$$
for $k \in \mathrm{SO}(2)$. Every function on $\mathcal{M}_2$ arises this way and $h^\phi \in C_c^\infty(\mathcal{M}_2)$ iff $\psi \in C_c^\infty(\mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R})$.

It is easy to go between the two parameterizations. In fact, the two parameterizations are linked via the isomorphism $\mathbb{H} \times \mathbb{R}^+ \simeq \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}$ given by $(u + iy, \lambda) \mapsto (\lambda y, \lambda/y, u)$. So given $\phi_0 \in C_c^\infty(\mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R})$, we have that $h_{\phi_0} = h_{\psi_0}$, where the function $\psi_0 \in C_c^\infty(\mathbb{H} \times \mathbb{R}^+)$ is defined by
$$\psi_0(u + iy, \lambda) = \phi_0(\lambda y, \lambda/y, u).$$

Let $\psi_1 \in C_c^\infty(\mathbb{H})$, $\psi_2 \in C_c^\infty(\mathbb{R}^+)$. We define $\psi_1 \times \psi_2 \in C_c^\infty(\mathbb{H} \times \mathbb{R}^+)$ to be the product function given by $\psi(z, \lambda) = \psi_1(z)\psi_2(\lambda)$ so that $h^{\psi_1 \times \psi_2} \in C_c^\infty(\mathcal{M}_2)$. Similarly, let $\phi_1 \in C_c^\infty(\mathbb{R}^+)$, $\phi_2 \in C_c^\infty(\mathbb{R}^+)$, $\phi_3 \in C_c^\infty(\mathbb{R})$, We define $\phi_1 \times \phi_2 \times \phi_3 \in C_c^\infty(\mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R})$ be the product function given by $(\phi_1 \times \phi_2 \times \phi_3)(t_1, t_2, u) = \phi_1(t_1)\phi_2(t_2)\phi_3(u)$ so that $h^{\phi_1 \times \phi_2 \times \phi_3} \in C_c^\infty(\mathcal{M}_2)$. We say that $h \in C_c^\infty(\mathcal{M}_2)$ is factorizable if it is of the form $h^{\psi_1 \times \psi_2}$ or $h^{\phi_1 \times \phi_2 \times \phi_3}$.

We define

- $R_1 := \{h^{\psi_1 \times \psi_2} : \psi_1 \in C_c^\infty(\mathbb{H}),\ \psi_2 \in C_c^\infty(\mathbb{R}^+)\}$,
- $R_2 := \{h^{\phi_1 \times \phi_2 \times \phi_3} : \phi_1 \in C_c^\infty(\mathbb{R}^+),\ \phi_2 \in C_c^\infty(\mathbb{R}^+),\ \phi_3 \in C_c^\infty(\mathbb{R})\}$.

Proposition 2.5. *For each $S \in \Lambda_2$, pick $r(S) \in \{1, 2\}$. Then the set of finite linear combinations of Poincaré series P_S^h with $S \in \Lambda_2$ and $h \in R_{r(S)}$ is dense in the space $C_c^\infty(\Gamma\backslash\mathbb{H}_2)$ equipped with the uniform topology.*

Proof. For $m = 1, 2$, let $\widetilde{R}_m \subset C_c^\infty(\mathcal{M}_2)$ be the set of finite linear combinations of elements of R_m. In view of Proposition 2.3, it suffices to show given $h_1 \in C_c^\infty(\mathcal{M}_2)$, $S \in \Lambda_2$, and $m \in \{1, 2\}$, there exists $h_2 \in \widetilde{R}_m$ such that

$$|P_S^{h_1}(Z) - P_S^{h_2}(Z)| < \varepsilon, \text{ for all } Z \in \Gamma\backslash\mathbb{H}_2. \tag{2.13}$$

To show (2.13), we let C be a compact set containing the support of h_1 and we choose C' to be a compact set whose interior contains C. By enlarging C and C' if needed, we can and will assume that they are both products of compact sets:

- If $m = 1$, then $C = C_1 \times C_2$, $C' = C_1' \times C_2'$ where $C_1 \subset C_1' \subset \mathbb{H}$ and $C_2 \subset C_2' \subset \mathbb{R}^+$ are compact.
- If $m = 2$, then $C = C_1 \times C_2 \times C_3$, $C' = C_1' \times C_2' \times C_3'$ where $C_1 \subset C_1' \subset \mathbb{R}^+$ and $C_2 \subset C_2' \subset \mathbb{R}^+$ and $C_3 \subset C_3' \subset \mathbb{R}$ are all compact.

Let $N_{C'}$ be as in Lemma 2.2. By applying the Stone–Weierstrass theorem on the algebra of smooth factorizable functions on C, we see that the uniform span of such functions contain h_1. By noting that any smooth factorizable function on C can be smoothly extended to an element of $\widetilde{R}_m$ with support in C', it follows that there exists $h_2 \in \widetilde{R}_m$ such that h_2 is supported on C' and $\sup |h_2 - h_1| < \frac{\varepsilon}{N_{C'}}$.

Now, using (2.3) and Lemma 2.2, it follows that $|P_S^{h_1}(Z) - P_S^{h_2}(Z)| < N_{C'} \cdot \frac{\varepsilon}{N_{C'}} = \varepsilon$, which completes the proof of (2.13). □

Corollary 2.6. *Let F_i be a sequence consisting of elements in $S_k(\Gamma)$. For each $S \in \Lambda_2$, let $r(S) \in \{1, 2\}$. Suppose that for each $S \in \Lambda_2$ and $h \in R_{r(S)}$, we have $D_{F_i}(P_S^h) \longrightarrow 0$ as $i \longrightarrow \infty$. Then for each ϕ in $C_b(\Gamma\backslash\mathbb{H}_2)$ we have $D_{F_i}(\phi) \longrightarrow 0$ as $i \longrightarrow \infty$.*

Proof. This is immediate from Propositions 2.1 and 2.5. □

Remark 2. *In the next section, we will end up choosing $r(S) = 1$ if $S = 0$, and $r(S) = 2$ if $S \neq 0$.*

3. Reduction of the proof to estimates of shifted convolution sums

The goal of this section is to reduce the proof of Theorem 1.2 to two key results, Proposition 3.1 and Proposition 3.2, which will be proved in Sections 4 and 5, respectively.

3.1. **Preliminary reduction.** Let $F \in S_k(\Gamma)$ be a Saito–Kurokawa lift that is a Hecke eigenform. We note first that Theorem 1.2 will follow once we know the following two statements.

i) (The off-diagonal case). For fixed $\phi_1 \in C_c^\infty(\mathbb{R}^+)$, $\phi_2 \in C_c^\infty(\mathbb{R}^+)$, $\phi_3 \in C_c^\infty(\mathbb{R})$, $L = \begin{pmatrix} \ell_1 & \ell_2/2 \\ \ell_2/2 & \ell_3 \end{pmatrix} \in \Lambda_2$ with $(\ell_1, \ell_2, \ell_3) \neq (0, 0, 0)$ and $h = h^{\phi_1 \times \phi_2 \times \phi_3}$, we have

$$\frac{1}{\|F\|_2^2} \int\limits_{\Gamma\backslash\mathbb{H}_2} |F(Z)|^2 P_L^h(Z) (\det Y)^k \, \mathrm{d}\mu \longrightarrow 0, \quad \text{as } k \longrightarrow \infty. \tag{3.1}$$

ii) (The diagonal case). For fixed $\psi_1 \in C_c^\infty(\mathbb{H})$, $\psi_2 \in C_c^\infty(\mathbb{R}^+)$ and $h = h^{\psi_1 \times \psi_2}$, we have

$$\frac{1}{\|F\|_2^2} \int\limits_{\Gamma\backslash\mathbb{H}_2} |F(Z)|^2 P_0^h(Z) (\det Y)^k \, \mathrm{d}\mu \longrightarrow \frac{1}{\mathrm{vol}(\Gamma\backslash\mathbb{H}_2)} \int\limits_{\Gamma\backslash\mathbb{H}_2} P_0^h(Z) \, \mathrm{d}\mu, \quad \text{as } k \longrightarrow \infty. \tag{3.2}$$

Proof of Theorem 1.2 assuming (3.1) *and* (3.2). This is an immediate consequence of Corollary 2.6 together with the observation that whenever $L = \begin{pmatrix} \ell_1 & \ell_2/2 \\ \ell_2/2 & \ell_3 \end{pmatrix} \neq 0$ we have

$$\int\limits_{\Gamma\backslash\mathbb{H}_2} P_L^h(Z) \, \mathrm{d}\mu = \int\limits_{\Gamma_\infty\backslash\mathbb{H}_2} e(\mathrm{Tr}(L\mathrm{Re}(Z))) h(\iota^{-1}(\mathrm{Im}(Z))) \, \mathrm{d}\mu = 0.$$

□

3.2. **Some properties of Saito–Kurokawa lifts.** Let $F \in S_k(\Gamma)$. It has a Fourier expansion

$$F(Z) = \sum_{S \in \Lambda_2} a(S) e(\mathrm{Tr}(SZ)),$$

where $a(S) = 0$ unless $S \in \Lambda_2^+$. We have the relation

$$a(T) = (\det A)^k a({}^tATA)$$

for $A \in \mathrm{GL}_2(\mathbb{Z})$. In particular, the Fourier coefficient $a(T)$ depends only on the $\mathrm{SL}_2(\mathbb{Z})$-equivalence class of T. We define the Petersson norm $\|F\|_2$ via

$$\|F\|_2^2 = \int\limits_{\Gamma\backslash\mathbb{H}_2} |F(Z)|^2 (\det Y)^k \mathrm{d}\mu.$$

For each $T \in \Lambda_2$, we let

$$R(T) := |\mathrm{disc}(T)|^{-k/2+3/4} a(T)$$

denote the normalized Fourier coefficient.

Now, suppose that $F \in S_k(\Gamma)$ is a *Saito–Kurokawa lift* and a Hecke eigenform. Then k is even and there exists $\widetilde{f} \in S_{k-\frac{1}{2}}(\Gamma_0(4))$ which is a classical half-integral weight form that F is lifted from [EZ85, §6]. It is known that $\widetilde{f}$ is a newform and lies in the Kohnen plus space. Precisely, if $\widetilde{f}$ has the Fourier expansion

$$\widetilde{f}(z) = \sum_{n \equiv 0,3\,(4)} c(n) n^{k/2-3/4} e(nz)$$

then the normalized Fourier coefficients of F and $\widetilde{f}$ are related by

$$R(T) = \sum_{j | \mathrm{cont}(T)} \sqrt{j}\, c\left(\tfrac{|\mathrm{disc}(T)|}{j^2}\right). \tag{3.3}$$

We let $f \in S_{2k-2}(\mathrm{SL}_2(\mathbb{Z}))$ be the normalized Hecke eigenform associated to $\widetilde{f}$ via the Shimura correspondence. Define $\langle f, f\rangle = \int_{\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}} |f(z)|^2 y^{2k-2} \frac{dxdy}{y^2}$. We let Π be the automorphic representation of $\mathrm{GSp}_4(\mathbb{A})$ attached to F and we let π_0 be the automorphic representation of $\mathrm{GL}_2(\mathbb{A})$ attached to f. From the characterization of Saito–Kurokawa lifts as CAP representations, one has the relation (which we will not need to use)

$$L(s, \Pi) = L(s, \pi_0)\zeta(s + 1/2)\zeta(s - 1/2),$$

where $L(s, \Pi)$ is the (finite part of the) degree 4 L-function attached to Π and $L(s, \pi_0)$ is the finite part of the degree 2 L-function attached to π_0.

We have the following key relation between the Petersson norms of F and $\widetilde{f}$ (see, e.g., [Bro07])

$$\frac{\|\widetilde{f}\|_2^2}{\|F\|_2^2} = \frac{24\ \pi^k}{\Gamma(k) L(\frac{3}{2}, \pi_0)}, \tag{3.4}$$

where

$$\|\widetilde{f}\|_2^2 = \frac{1}{6} \int\limits_{\Gamma_0(4)\backslash\mathbb{H}} |\widetilde{f}(z)|^2 y^{k-\frac{1}{2}} \frac{dxdy}{y^2}.$$

On the other hand, by Waldspurger's formula [KZ81], we have for each negative fundamental discriminant d that

$$\frac{|c(|d|)|^2}{\|\widetilde{f}\|_2^2} = \frac{L(\frac{1}{2}, \pi_0 \otimes \chi_d)}{\langle f, f\rangle} \frac{\Gamma(k-1)}{\pi^{k-1}} = \frac{L(\frac{1}{2}, \pi_0 \otimes \chi_d)}{L(1, \mathrm{sym}^2 \pi_0)} \frac{2^{2k-2}\pi^{k+\frac{1}{2}}}{\Gamma(k-\frac{1}{2})}, \tag{3.5}$$

where the second equality uses the duplication formula for the Gamma function and the well-known relation (see, e.g., [Nel11, (7)])

$$\langle f, f\rangle = 2^{-4k+5}\pi^{-2k+1}\Gamma(2k-2) L(1, \mathrm{sym}^2 \pi_0) = \frac{24\pi c_k}{(k-1)} L(1, \mathrm{sym}^2 \pi_0),$$

where

$$c_k := \frac{\Gamma(k)\Gamma\left(k - \frac{1}{2}\right)}{3 \cdot 2^{2k+1} \cdot \pi^{2k+\frac{1}{2}}}. \tag{3.6}$$

Combining (3.3), (3.4) and (3.5), we see that for each $T \in \Lambda_2$ such that $\mathrm{disc}(T) = d$ is a negative fundamental discriminant, we have

$$c_k \frac{|R(T)|^2}{\|F\|_2^2} = \frac{L(\frac{1}{2}, \pi_0 \otimes \chi_d)}{L(1, \mathrm{sym}^2 \pi_0) L(\frac{3}{2}, \pi_0)}. \tag{3.7}$$

3.3. **Main results on shifted convolution sums.** Let $F \in S_k(\Gamma)$ be a Saito–Kurokawa lift that is a Hecke eigenform. We let $R(T)$, $T \in \Lambda_2$ denote the normalized Fourier coefficients of F, as defined above. Let c_k be given by (3.6). The conditions (3.1) and (3.2) reduce to estimates on shifted convolution sums involving the Fourier coefficients $R(T)$. In particular, as we will show later in this section, they will be implied by the following key propositions.

Proposition 3.1. *Let $F \in S_k(\Gamma)$ be a Saito–Kurokawa lift that is a Hecke eigenform and let $R(T)$ denote the normalized Fourier coefficients of F. Fix $h_1 \in C_c^\infty(\mathbb{R}^+)$, $h_2 \in C_c^\infty(\mathbb{R}^+)$, $h_3 \in C_c^\infty(\mathbb{R})$, $L = \begin{pmatrix} \ell_1 & \ell_2/2 \\ \ell_2/2 & \ell_3 \end{pmatrix} \in \Lambda_2$ with $(\ell_1, \ell_2, \ell_3) \neq (0,0,0)$. Assume GRH. Then for any $\varepsilon > 0$,*

$$\frac{c_k}{\|F\|_2^2} \sum_{T = \begin{pmatrix} m & r/2 \\ r/2 & n \end{pmatrix} \in \Lambda_2} |R(T)R(T+L)|\, h_1\left(\frac{m}{k}\right) h_2\left(\frac{n}{k}\right) h_3\left(\frac{r}{k}\right) \ll_{h_i, L, \varepsilon} \frac{k^3}{(\log k)^{1/28-\varepsilon}}.$$

We will prove Proposition 3.1 in Section 4.

Next, for each slowly growing function $g : \mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H} \longrightarrow \mathbb{C}$, $\kappa \in C_c^\infty(\mathbb{R}^+)$, and $T \in \Lambda_2$, define the following quantities which depend only on the class of T in $\Lambda_2/\mathrm{SL}_2(\mathbb{Z})$:

$$\varepsilon(T) := |\{A \in \mathrm{SL}_2(\mathbb{Z}) : {}^tATA = T\}|$$

and

$$G(T; g, \kappa) := \int_{\substack{\lambda > 0 \\ z = u+iy \in \mathbb{H}}} g(z) \lambda^{2k} \kappa(\lambda) e^{-4\pi\lambda \mathrm{Tr}(T g_z {}^t g_z)} \frac{\mathrm{d}u\, \mathrm{d}y\, \mathrm{d}\lambda}{y^2 \lambda^4}, \tag{3.8}$$

where for $z = u + iy$ we write $g_z := \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \begin{pmatrix} y^{\frac{1}{2}} & 0 \\ 0 & y^{-\frac{1}{2}} \end{pmatrix}$. Note that g_z takes the point i to the point z and hence for each $A \in \mathrm{SL}_2(\mathbb{R})$ we have $g_{Az}^{-1} A g_z \in SO(2)$.

Proposition 3.2. *Let $F \in S_k(\Gamma)$ be a Saito–Kurokawa lift that is a Hecke eigenform and let $R(T)$ denote the normalized Fourier coefficients of F. Fix $g \in C_c^\infty(\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H})$ and $\kappa \in C_c^\infty(\mathbb{R}^+)$. Assume GRH. Then as $k \longrightarrow \infty$,*

$$\frac{1}{\|F\|_2^2} \sum_{T \in \Lambda_2/\mathrm{SL}_2(\mathbb{Z})} \frac{|R(T)|^2}{\varepsilon(T)} |\mathrm{disc}(T)|^{k-\frac{3}{2}} G(T; g, \kappa) \longrightarrow \frac{\widetilde{\kappa}(3)}{2\mathrm{vol}(\Gamma\backslash\mathbb{H}_2)} \int_{\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}} g(u+iy) \frac{\mathrm{d}u\, \mathrm{d}y}{y^2}. \tag{3.9}$$

We will prove Proposition 3.2 in Section 5.

3.4. **The off-diagonal case.** In this subsection we show that Proposition 3.1 implies (3.1).

Lemma 3.3. *Fix $\phi_1 \in C_c^\infty(\mathbb{R}^+)$, $\phi_2 \in C_c^\infty(\mathbb{R}^+)$, $\phi_3 \in C_c^\infty(\mathbb{R})$, $L = \begin{pmatrix} \ell_1 & \ell_2/2 \\ \ell_2/2 & \ell_3 \end{pmatrix} \in \Lambda_2$ with $(\ell_1, \ell_2, \ell_3) \neq (0,0,0)$. Let $h = h^{\phi_1 \times \phi_2 \times \phi_3}$. Assume GRH and assume the truth of Proposition 3.1. Then* (3.1) *holds in the stronger form*

$$\frac{1}{\|F\|_2^2} \int_{\Gamma\backslash\mathbb{H}_2} |F(Z)|^2 P_L^h(Z) (\det Y)^k \, \mathrm{d}\mu \ll_{\phi_i, L, \varepsilon} (\log k)^{-1/28+\varepsilon}.$$

Proof. By unfolding we obtain

(3.10)

$$\begin{aligned}
&\frac{1}{\|F\|_2^2}\int\limits_{\Gamma\backslash\mathbb{H}_2} |F(Z)|^2 P_L^h(Z)(\det Y)^k \,\mathrm{d}\mu \\
&= \frac{1}{\|F\|_2^2}\int\limits_{\Gamma_\infty\backslash\mathbb{H}_2} |F(Z)|^2 h_L(Z)(\det Y)^k \,\mathrm{d}\mu \\
&= \frac{1}{\|F\|_2^2}\int\limits_{\Gamma_\infty\backslash\mathbb{H}_2} \left(\sum_{S_1\in\Lambda_2}\sum_{S_2\in\Lambda_2} a(S_1)\overline{a(S_2)} e(\mathrm{Tr}(S_1X)-\mathrm{Tr}(S_2X)) e^{-2\pi\mathrm{Tr}((S_1+S_2)Y)}\right) h_L(Z)(\det Y)^k \,\mathrm{d}\mu.
\end{aligned}$$

The space $\Gamma_\infty\backslash\mathbb{H}_2$ may be parameterized by the points $(n(X)m(A))\cdot iI_2$ with $X := \begin{pmatrix} x_1 & x_2 \\ x_2 & x_3\end{pmatrix}$, $x_i \in \mathbb{Z}\backslash\mathbb{R}$, and $A = \begin{pmatrix} 1 & u \\ & 1\end{pmatrix}\begin{pmatrix}\sqrt{t_1} & \\ & \sqrt{t_2}\end{pmatrix}$ with $t_i \in \mathbb{R}^+$ and $u \in \mathbb{R}$. Note also that we may write $n(X)m(A))\cdot iI_2 = X + iY_{t_1,t_2,u}$, where we have set $Y_{t_1,t_2,u} := \begin{pmatrix} t_1+t_2u^2 & t_2u \\ t_2u & t_2\end{pmatrix}$. Under the substitution $X+iY \mapsto X+iY_{t_1,t_2,u}$ the measure $d\mu$ is replaced by $t_1^{-3}t_2^{-2}\mathrm{d}x_1\,\mathrm{d}x_2\,\mathrm{d}x_3\,\mathrm{d}t_1\,\mathrm{d}t_2\,\mathrm{d}u$. Finally, we have $h_L(X+iY_{t_i,u}) = e(\mathrm{Tr}(LX))h(\iota^{-1}(Y_{t_i,u})) = e(\mathrm{Tr}(LX))\phi_1(t_1)\phi_2(t_2)\phi_3(u)$.

Therefore, after making the above substitutions and executing the $\mathrm{d}x_i$ integrals, (3.10) reduces to

$$\frac{1}{\|F\|_2^2}\sum_{T\in\Lambda_2} |\mathrm{disc}(T)\mathrm{disc}(T+L)|^{k/2-3/4} R(T)R(T+L)W_{h,L}(T), \tag{3.11}$$

where for $T = \begin{pmatrix} m & r/2 \\ r/2 & n\end{pmatrix}$ we have

$$W_{h,L}(T) := \int\limits_{\mathbb{R}}\int\limits_{\mathbb{R}^+}\int\limits_{\mathbb{R}^+} e^{-2\pi((2m+\ell_1)(t_1+u^2t_2)+(2r+\ell_2)t_2u+(2n+\ell_3)t_2)}\phi_1(t_1)\phi_2(t_2)\phi_3(u)\frac{\mathrm{d}t_1\,\mathrm{d}t_2\,\mathrm{d}u}{t_1^{3-k}t_2^{2-k}}.$$

We proceed to study the integral $W_{h,L}(T)$ in detail. For simplicity we set $v_1 := 2m+\ell_1$, $v_2 := 2r+\ell_2$, and $v_3 := 2n+\ell_3$ so that the integral takes the form

$$\int\limits_{\mathbb{R}}\int\limits_{\mathbb{R}^+}\int\limits_{\mathbb{R}^+} e^{-2\pi(v_1(t_1+u^2t_2)+v_2t_2u+v_3t_2))}\phi_1(t_1)\phi_2(t_2)\phi_3(u)\frac{\mathrm{d}t_1\,\mathrm{d}t_2\,\mathrm{d}u}{t_1^{3-k}t_2^{2-k}}.$$

Note that automatically $v_1, v_2 > 0$. Let us first treat the t_1-integral. Using Mellin inversion we compute

$$\begin{aligned}
\int\limits_0^\infty e^{-2\pi v_1t_1}t_1^{k-3}\phi_1(t_1)dt_1 &= \frac{1}{2\pi i}\int\limits_0^\infty\int\limits_{(2)} e^{-2\pi v_1t_1}t_1^{k+s-3}\widetilde{\phi_1}(s)\mathrm{d}s\,\mathrm{d}t_1 \\
&= \frac{1}{2\pi i}\int\limits_{(2)}(2\pi v_1)^{-s-k+2}\widetilde{\phi_1}(s)\Gamma(s+k-2)\mathrm{d}s \\
&= \frac{\Gamma(k-2)}{2\pi i}\int\limits_{(2)}(2\pi v_1)^{-s-k+2}\widetilde{\phi_1}(s)\frac{\Gamma(s+k-2)}{\Gamma(k-2)}\mathrm{d}s. \qquad (3.12)
\end{aligned}$$

By Stirling's formula,

$$\frac{\Gamma(s+k-2)}{\Gamma(k-2)} = (k-2)^s \left(1+R(s)\right),$$

where $R(s)$ is holomorphic in the half-plane $\mathrm{Re}(s) > -k+2$ and satisfies the estimate $R(s) \ll (|s|+1)^2/k$ there. Plugging this into (3.12) and shifting contours as far left as possible to handle the integral involving $R(s)$ using the decay properties of the Mellin transform, the t_1-integral is given by

$$\frac{\Gamma(k-2)}{2\pi i}(2\pi v_1)^{-k+2}\left(\int\limits_{(2)} \widetilde{\phi_1}(s)\left(\frac{k-2}{2\pi v_1}\right)^s \mathrm{d}s + O_{A,\phi_1,\varepsilon}\left(\frac{1}{k^{1-\varepsilon}}\cdot\frac{1}{1+(v_1/k)^A}\right)\right)$$
$$= \Gamma(k-2)(2\pi v_1)^{-k+2}\phi_1\left(\frac{k-2}{2\pi v_1}\right) + O_{A,\phi_1,\varepsilon}\left(\frac{\Gamma(k-2)(2\pi v_1)^{-k+2}}{k^{1-\varepsilon}}\cdot\frac{1}{1+(v_1/k)^A}\right)$$

for any $A > 1$.

Next we evaluate the u-integral. By Fourier inversion we compute

$$\begin{aligned}\int\limits_{\mathbb{R}} e^{-2\pi t_2(v_1u^2+v_2u)}\phi_3(u)\,\mathrm{d}u &= \int\limits_{\mathbb{R}} e^{-2\pi t_2(v_1u^2+v_2u)}\int\limits_{\mathbb{R}}\widehat{\phi}_3(y)e(uy)\mathrm{d}y\,\mathrm{d}u\\ &= \int\limits_{\mathbb{R}}\widehat{\phi}_3(y)\int\limits_{\mathbb{R}} e^{-2\pi t_2(v_1u^2+v_2u)}e(uy)\mathrm{d}u\,\mathrm{d}y\\ &= \int\limits_{\mathbb{R}}\widehat{\phi}_3(y)\cdot\frac{1}{\sqrt{t_2v_1}}e^{\frac{\pi}{2}\cdot\frac{(iy-v_2t_2)^2}{t_2v_1}}\mathrm{d}y\\ &= \frac{1}{\sqrt{2t_2v_1}}e^{\frac{\pi}{2}\cdot\frac{v_2^2t_2}{v_1}}\int\limits_{\mathbb{R}}\widehat{\phi}_3(y)e^{-\pi i\frac{v_2}{v_1}y}e^{-\frac{\pi y^2}{2t_2v_1}}\,\mathrm{d}y.\end{aligned}$$

Observe that

$$\int\limits_{\mathbb{R}}\widehat{\phi}_3(y)e^{-\pi i\frac{v_2y}{v_1}}\,\mathrm{d}y = \phi_3\left(-\frac{v_2}{2v_1}\right)$$

and $e^{-t} = 1+O(t)$ as $t\longrightarrow 0$. Using these the integral above takes the form

$$\frac{1}{\sqrt{2t_2v_1}}e^{\frac{\pi}{2}\cdot\frac{v_2^2t_2}{v_1}}\left(\phi_3\left(-\frac{v_2}{2v_1}\right)+E\right),$$

where

$$E \ll \frac{1}{t_2v_1}\int\limits_{\mathbb{R}}|\widehat{\phi}_3(y)|y^2\,\mathrm{d}y \ll_{\phi_3} \frac{1}{t_2v_1}$$

using the fact that ϕ_3 is a Schwartz function.

The remaining t_2-integral can be computed similarly as the t_1-integral:

$$\int\limits_0^\infty e^{-2\pi(v_3t_2-\frac{v_2^2}{4v_1}t_2)}t_2^{k-5/2}\phi_2(t_2)\,\mathrm{d}t_2 = \frac{1}{2\pi i}\int\limits_{(2)}\int\limits_0^\infty e^{-2\pi(v_3t_2-\frac{v_2^2}{4v_1}t_2)}t_2^{s+k-5/2}\widetilde{\phi_2}(s)\,\mathrm{d}t_2\,\mathrm{d}s$$
$$= \frac{\Gamma\left(k-\frac{3}{2}\right)}{2\pi i}\int\limits_{(2)}\left(2\pi\left(v_3-\frac{v_2^2}{4v_1}\right)\right)^{-s-k+3/2}\widetilde{\phi_2}(s)\frac{\Gamma\left(s+k-\frac{3}{2}\right)}{\Gamma\left(k-\frac{3}{2}\right)}\,\mathrm{d}s.$$

Using Stirling's formula for $\Gamma(s+k-3/2)/\Gamma(k-3/2)$ and shifting contours, the t_2-integral becomes

$$\begin{aligned}
&\frac{\Gamma\left(k-\frac{3}{2}\right)}{2\pi i}\left(2\pi\left(v_3-\frac{v_2^2}{4v_1}\right)\right)^{-k+3/2}\left(\int\limits_{(2)}\left(\frac{k-\frac{3}{2}}{2\pi\left(v_3-\frac{v_2^2}{4v_1}\right)}\right)^s\widetilde{\phi_2}(s)\,\mathrm{d}s\right.\\
&\qquad\left.+O_{A,\phi_2,\varepsilon}\left(\frac{1}{k^{1-\varepsilon}}\cdot\frac{1}{1+((v_3-\frac{v_2^2}{4v_1})/k)^A}\right)\right)\\
&=\Gamma\left(k-\frac{3}{2}\right)\left(2\pi\left(v_3-\frac{v_2^2}{4v_1}\right)\right)^{-k+3/2}\phi_2\left(\frac{k-\frac{3}{2}}{2\pi\left(v_3-\frac{v_2^2}{4v_1}\right)}\right)\\
&\qquad+O_{A,\phi_2,\varepsilon}\left(\Gamma\left(k-\frac{3}{2}\right)\left(2\pi\left(v_3-\frac{v_2^2}{4v_1}\right)\right)^{-k+3/2}\frac{1}{k^{1-\varepsilon}}\cdot\frac{1}{1+\left(\left(v_3-\frac{v_2^2}{4v_1}\right)/k\right)^A}\right)
\end{aligned}$$

for any $A>1$.

The contribution from the error E can be estimated by a similar reasoning to see that it contributes $\ll\Gamma(k-3)(2\pi k)^{-k+3}$, which is sufficiently small.

Putting the above computations together we see that the weight function $W_{h,L}(T)$ equals

$$\begin{aligned}
&\frac{(2\pi)^{-2k+7/2}\Gamma(k-2)\Gamma\left(k-\frac{3}{2}\right)}{\sqrt{2}((2m+\ell_1)(2n+\ell_3)-(r+\ell_2/2)^2)^{k-3/2}}\\
&\qquad\times\phi_1\left(\frac{k-2}{2\pi(2m+\ell_1)}\right)\phi_3\left(-\frac{2m+\ell_1}{2(2r+\ell_2)}\right)\phi_2\left(\frac{k-\frac{3}{2}}{2\pi\left((2n+\ell_3)-\frac{(2r+\ell_2)^2}{4(2m+\ell_1)}\right)}\right)\left(1+O_{\phi_i,\varepsilon}\left(k^{-1+\varepsilon}\right)\right)
\end{aligned}$$

for $T=\begin{pmatrix}m & r/2\\ r/2 & n\end{pmatrix}\in\Lambda_2$.

From the compact support of the functions ϕ_i, it is clear that there exist compact subsets $C_1\subset\mathbb{R}^+$, $C_2\subset\mathbb{R}^+$, $C_3\subset\mathbb{R}$ depending on the ϕ_i and the ℓ_i so that for all sufficiently large k, we have that

$$\phi_1\left(\frac{k-2}{2\pi(2m+\ell_1)}\right)\phi_3\left(-\frac{2m+\ell_1}{2(2r+\ell_2)}\right)\phi_2\left(\frac{k-\frac{3}{2}}{2\pi\left((2n+\ell_3)-\frac{(2r+\ell_2)^2}{4(2m+\ell_1)}\right)}\right)\neq 0$$

implies $\frac{m}{k}\in C_1$, $\frac{n}{k}\in C_2$, and $\frac{r}{k}\in C_3$.

Pick non-negative valued functions $h_1\in C_c^\infty(\mathbb{R}^+)$, $h_2\in C_c^\infty(\mathbb{R}^+)$, $h_3\in C_c^\infty(\mathbb{R})$, such that $h_i(x)=\|\phi_i\|_\infty$ for $x\in C_i$. We have

$$\left|\phi_1\left(\tfrac{k-2}{2\pi(2m+\ell_1)}\right)\phi_3\left(-\tfrac{2m+\ell_1}{2(2r+\ell_2)}\right)\phi_2\left(\frac{k-\frac{3}{2}}{2\pi\left((2n+\ell_3)-\frac{(2r+\ell_2)^2}{4(2m+\ell_1)}\right)}\right)\right|\leq h_1\left(\tfrac{m}{k}\right)h_2\left(\tfrac{n}{k}\right)h_3\left(\tfrac{r}{k}\right).$$

Therefore

$$\begin{aligned}
|W_{h,L}(T)|&\ll\frac{(2\pi)^{-2k}\Gamma(k-2)\Gamma\left(k-\frac{3}{2}\right)}{((2m+\ell_1)(2n+\ell_3)-(r+\ell_2/2)^2)^{k-3/2}}h_1\left(\frac{m}{k}\right)h_2\left(\frac{n}{k}\right)h_3\left(\frac{r}{k}\right)\\
&\ll\frac{c_k}{k^3\ |\mathrm{disc}(T)\mathrm{disc}(T+L)|^{k/2-3/4}}\ h_1\left(\frac{m}{k}\right)h_2\left(\frac{n}{k}\right)h_3\left(\frac{r}{k}\right)
\end{aligned}$$

and consequently (3.11) is

$$\ll \frac{c_k}{k^3\ \|F\|_2^2} \sum_{T=\begin{pmatrix} m & r/2 \\ r/2 & n \end{pmatrix} \in \Lambda_2} |R(T)R(T+L)| h_1\left(\frac{m}{k}\right) h_2\left(\frac{n}{k}\right) h_3\left(\frac{r}{k}\right) \ll_{h_i,\varepsilon} (\log k)^{-1/28+\varepsilon},$$

where the last estimate follows from Proposition 3.1. □

3.5. **The diagonal case.** In this subsection we show that Proposition 3.2 implies (3.2).

Lemma 3.4. *Fix $\psi_1 \in C_c^\infty(\mathbb{H})$, $\psi_2 \in C_c^\infty(\mathbb{R}^+)$ and let $h = h^{\psi_1 \times \psi_2}$. Assume GRH and assume the truth of Proposition 3.2. Then we have*

$$\frac{1}{\|F\|_2^2} \int\limits_{\Gamma\backslash\mathbb{H}_2} |F(Z)|^2 P_0^h(Z) (\det Y)^k d\mu \longrightarrow \frac{1}{\mathrm{vol}(\Gamma\backslash\mathbb{H}_2)} \int\limits_{\Gamma\backslash\mathbb{H}_2} P_0^h(Z)\, d\mu \ \text{ as } \ k \longrightarrow \infty. \tag{3.13}$$

Proof. Recall that for $g \in G$,

$$P_0^h(g) = \sum_{\gamma \in \Gamma_\infty\backslash\Gamma} h_0(\gamma g) = \sum_{\gamma \in P(\mathbb{Z})\backslash\Gamma} \sum_{\eta \in \mathrm{SL}_2(\mathbb{Z})} h_0(m(\eta)\gamma g) = \sum_{\gamma \in P(\mathbb{Z})\backslash\Gamma} h_0'(\gamma g),$$

where we define the function h_0' on G via

$$h_0'(g) := \sum_{\eta \in \mathrm{SL}_2(\mathbb{Z})} h_0(m(\eta) g).$$

Since h_0' is right K_∞-invariant, it defines a function on $G/K_\infty \simeq \mathbb{H}_2$ which we also denote as h_0'. Therefore, for $Z \in \mathbb{H}_2$, we have $P_0^h(Z) = \sum_{\gamma \in P(\mathbb{Z})\backslash\Gamma} h_0'(\gamma Z)$. The space $P(\mathbb{Z})\backslash\mathbb{H}_2$ may be parametrized by the points $(n(X)m(A)) \cdot iI_2$ with $X = \begin{pmatrix} x_1 & x_2 \\ x_2 & x_3 \end{pmatrix}$, $x_i \in \mathbb{Z}\backslash\mathbb{R}$, and $A = \begin{pmatrix} 1 & u \\ & 1 \end{pmatrix} \begin{pmatrix} y^{1/2}\lambda^{1/2} & \\ & y^{-1/2}\lambda^{1/2} \end{pmatrix}$ with $\lambda \in \mathbb{R}^+$ and $u + iy \in \mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}$. Note also that we may write $n(X)m(A)) \cdot iI_2 = X + i\lambda Y_{u,y}$, where we have set $Y_{u,y} := \begin{pmatrix} y + u^2/y & u/y \\ u/y & 1/y \end{pmatrix}$. Under the substitution $X + iY \mapsto X + i\lambda Y_{u,y}$ the measure $d\mu$ is replaced by $2y^{-2}\lambda^{-4} \mathrm{d}x_1\, \mathrm{d}x_2\, \mathrm{d}x_3\, \mathrm{d}y\, \mathrm{d}\lambda\, \mathrm{d}u$. Finally, an easy calculation shows that $h_0'(X + i\lambda Y_{u,y}) = g_{\psi_1}(u+iy)\psi_2(\lambda)$, where the function $g_{\psi_1} : \mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H} \longrightarrow \mathbb{C}$ is defined by

$$g_{\psi_1}(z) := \sum_{\eta \in \mathrm{SL}_2(\mathbb{Z})} \psi_1(\eta z).$$

Therefore, by unfolding we have

(3.14)

$$\frac{1}{\|F\|_2^2} \int\limits_{\Gamma\backslash\mathbb{H}_2} |F(Z)|^2 P_0^h(Z) (\det Y)^k \,\mathrm{d}\mu = \frac{1}{\|F\|_2^2} \int\limits_{P(\mathbb{Z})\backslash\mathbb{H}_2} |F(Z)|^2 h_0'(Z) (\det Y)^k \,\mathrm{d}\mu$$

$$= \frac{2}{\|F\|_2^2} \int\limits_{u+iy \in \mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}} \int\limits_{\lambda>0} \left(\sum_{S \in \Lambda_2} |a(S)|^2 e^{-4\pi\lambda \mathrm{Tr}(SY_{u,y})} \right) g_{\psi_1}(u+iy)\psi_2(\lambda)\lambda^{2k-4} y^{-2} \,\mathrm{d}u\, \mathrm{d}y\, \mathrm{d}\lambda.$$

(3.15)

Recall that $a(S) = a({}^tASA)$ for $A \in \mathrm{SL}_2(\mathbb{Z})$. For $z = u + iy$, write $g_z = \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}\begin{pmatrix} y^{\frac{1}{2}} & 0 \\ 0 & y^{-\frac{1}{2}} \end{pmatrix}$, so that $Y_{u,y} = g_z {}^t g_z$. Note that $\mathrm{Tr}({}^tASAY_{u,y}) = \mathrm{Tr}(SAg_z {}^t g_z {}^t A) = \mathrm{Tr}(Sg_{Az} {}^t g_{Az})$. Recall that $\varepsilon(T) := |\{A \in \mathrm{SL}_2(\mathbb{Z}) : {}^tATA = T\}|$. We see that (3.15) equals

$$\frac{2}{\|F\|_2^2} \int\limits_{z \in \mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}} \sum_{A \in \mathrm{SL}_2(\mathbb{Z})} \int\limits_{\lambda > 0} \left(\sum_{S \in \Lambda_2/\mathrm{SL}_2(\mathbb{Z})} \frac{|a(S)|^2}{\varepsilon(S)} e^{-4\pi\lambda \mathrm{Tr}(Sg_{Az} {}^t g_{Az})} \right) g_{\psi_1}(Az)\psi_2(\lambda) \frac{\lambda^{2k-4}}{y^2} \, \mathrm{d}u \, \mathrm{d}y \, \mathrm{d}\lambda$$
$$= \frac{4}{\|F\|_2^2} \sum_{T \in \Lambda_2/\mathrm{SL}_2(\mathbb{Z})} \frac{|R(T)|^2}{\varepsilon(T)} |\mathrm{disc}(T)|^{k-\frac{3}{2}} G(T; g_{\psi_1}, \psi_2),$$

where in the last step we use the fact that $\mathrm{SL}_2(\mathbb{Z})/\{\pm 1\}$ acts simply transitively on $\mathbb{H}_2$. On the other hand, we have that

$$\begin{aligned} \int\limits_{\Gamma\backslash\mathbb{H}_2} P_0^h(Z) \, \mathrm{d}\mu &= \int\limits_{P(\mathbb{Z})\backslash\mathbb{H}_2} h_0'(Z) \mathrm{d}\mu \\ &= 2 \int\limits_{u+iy \in \mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}} \int\limits_{\lambda > 0} g_{\psi_1}(u+iy)\psi_2(\lambda) \frac{\mathrm{d}u\mathrm{d}y\mathrm{d}\lambda}{y^2\lambda^4} \\ &= 2 \, \widetilde{\psi_2}(3) \int\limits_{\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}} g_{\psi_1}(u+iy) \frac{\mathrm{d}u\mathrm{d}y}{y^2}. \end{aligned}$$

Therefore, using Proposition 3.2, we see that (3.13) holds. □

4. Proof of Proposition 3.1

Our next objective is to establish Proposition 3.1.

Let f be a weight $2k-2$ newform of fixed level[5] N. Let F_1, F_2, F_3 be Schwartz functions. Given a function $G : \mathbb{R}^2 \longrightarrow \mathbb{C}$, non-negative integers f_1, f_2, and integers ℓ_1, ℓ_2, ℓ_3 we write

$$\sum\nolimits' G(d_1, d_2) := \sum_{d_1, d_2 \in \mathbb{Z}}^{\flat} G(d_1, d_2) \sum_{\substack{r,m,n \in \mathbb{Z} \\ (r^2-4mn)/f_1^2 = d_1 \\ ((r+\ell_1)^2 - 4(m+\ell_2)(n+\ell_3))/f_2^2 = d_2}} F_1\left(\frac{r}{k}\right) F_2\left(\frac{m}{k}\right) F_3\left(\frac{n}{k}\right),$$

where $\sum^{\flat}$ denotes that the sum over fundamental discriminants d_1, d_2. Given integers ℓ_1, ℓ_2, ℓ_3 write $l = \prod_{j:\ell_j \neq 0} \ell_j$. The following auxiliary result plays a key role in the proof of Proposition 3.1 and we shall establish this first.

Proposition 4.1. *Assume GRH. Let $(\ell_1, \ell_2, \ell_3) \in \mathbb{Z}^3 \backslash \{(0,0,0)\}$ and $f_1, f_2 \in \mathbb{N}$. Then for uniformly in $\ell_1, \ell_2, \ell_3, f_1, f_2$ we have that*

$$\sum\nolimits' \sqrt{L(\tfrac{1}{2}, f \otimes \chi_{d_1}) L(\tfrac{1}{2}, f \otimes \chi_{d_2})} \ll k^3 \exp\Big(\sum_{p | l f_1 f_2} \frac{1}{\sqrt{p}} \Big) \frac{1}{(\log k)^{1/4-\varepsilon}},$$

where the implied constant depends on N, F_1, F_2, F_3, and ε.

[5]In this section we work with general level, but our application only requires the case of the full level.

We assume GRH for $L(s, f\otimes\chi_d)$, for all fundamental discriminants, and $L(s, \mathrm{sym}^2 f)$. The argument to prove Proposition 4.1 uses Soundararajan's method [Sou09] for bounding moments of L-functions along with some of the techniques developed in [LR20], where a similar, yet simpler moment bound is required. We also require the following lemma [RlS15, Lemma 7].

Lemma 4.2. *Let F be a Schwartz function and $\eta \pmod q$ be a congruence class modulo q. Suppose n is an odd integer co-prime to q. Then*

$$\sum_{d\equiv\eta\,(q)}\left(\frac{d}{n}\right)F(d) = \frac{1}{qn}\left(\frac{q}{n}\right)\sum_{j\in\mathbb{Z}}\widehat{F}\left(\frac{j}{nq}\right)e\left(\frac{j\eta\overline{n}}{q}\right)\tau_j(n),$$

where for $j\in\mathbb{Z}$ and $n\in\mathbb{N}$ we have set

$$\tau_j(n) := \sum_{b\,(n)}\left(\frac{b}{n}\right)e\left(\frac{jn}{b}\right).$$

4.1. **The character sum.** To prove Proposition 4.1 we will need to estimate a certain intricate character sum. Let s,t be odd natural numbers with $s,t\le k^{1/2}$ and ℓ_1,ℓ_2,ℓ_3 be fixed integers. Let F_1,F_2,F_3 be Schwartz functions. Define P_s to be the set of primes dividing s but not t, P_t be the primes dividing t but not s, and $P_{s,t}$ be the primes dividing s,t. Also, let α_p be the non-negative integer such that $p^{\alpha_p}||s$ and β_p be the non-negative integer such that $p^{\beta_p}||t$. Given $G:\mathbb{R}^2\longrightarrow\mathbb{C}$ and integers ℓ_1,ℓ_2,ℓ_3 we use the notation

$$\sum\nolimits'' G(d_1,d_2) := \sum_{d_1,d_2\in\mathbb{Z}} G(d_1,d_2) \sum_{\substack{r,m,n\in\mathbb{Z}\\ r^2-4mn=d_1\\ (r+\ell_1)^2-4(m+\ell_2)(n+\ell_3)=d_2}} F_1\left(\frac{r}{k}\right)F_2\left(\frac{m}{k}\right)F_3\left(\frac{n}{k}\right),$$

where we allow d_1,d_2 to be any integers (not just fundamental discriminants).

Proposition 4.3. *Let F_1,F_2,F_3 be Schwartz functions. Then for odd $s,t\le k^{1/3}$ we have that*

$$\sum\nolimits''\left(\frac{d_1}{s}\right)\left(\frac{d_2}{t}\right) = k^3\widehat{F}_1(0)\widehat{F}_2(0)\widehat{F}_3(0)f(s,t)+O(k^{-100}), \tag{4.1}$$

where

$$\begin{aligned} f(s,t) := {} & \prod_{\substack{p\in P_s\\ 2|\alpha_p}}\left(1-\frac{1}{p}\right)\prod_{\substack{p\in P_s\\ 2\nmid\alpha_p}}\left(\frac{1}{p}-\frac{1}{p^2}\right)\prod_{\substack{p\in P_t\\ 2|\beta_p}}\left(1-\frac{1}{p}\right)\prod_{\substack{p\in P_t\\ 2\nmid\beta_p}}\left(\frac{1}{p}-\frac{1}{p^2}\right)\\ & \times\prod_{\substack{p\in P_{s,t}\\ 2|\alpha_p,\beta_p}}\ell_1(p)\prod_{\substack{p\in P_{s,t}\\ \alpha_p\not\equiv\beta_p\,(2)}}\ell_2(p)\prod_{\substack{p\in P_{s,t}\\ 2\nmid\alpha_p,\beta_p}}\ell_3(p) \end{aligned} \tag{4.2}$$

and

$$\ell_1(p) = 1+O\left(\frac{1}{p}\right),\qquad \ell_2(p)=O\left(\frac{1}{p}\right),\qquad \ell_3(p)=\begin{cases}1-\frac{1}{p}\ \text{if } p|\ell_j,\ \forall j\in\{1,2,3\},\\ O\left(\frac{1}{\sqrt{p}}\right)\ \text{otherwise.}\end{cases}$$

4.1.1. *Local sums.* We will make use of the following easy observation repeatedly.

Lemma 4.4. *Let p be an odd prime and d be a congruence class $\pmod p$. When a,b,c run over congruence classes $\pmod p$, a^2-4bc attains the value $d \pmod p$ for $p^2+(\frac{d}{p})p$ triples.*

Proof. The number of triples $(a,b,c)\in(\mathbb{Z}/p\mathbb{Z})^3$ with $a^2-4bc=d$ equals

$$\sum_{b,c\,(p)}\left(1+\left(\frac{d+4bc}{p}\right)\right).$$

Also, for $b \not\equiv 0\,(p)$ we have that

$$\sum_{c\,(p)} \left(\frac{d+4bc}{p}\right) = 0$$

and if $b \equiv 0\,(p)$ the sum above is clearly equal to $(\frac{d}{p})p$. Combining the two preceding estimates gives the claim. □

The first consequence of this is that

$$\sum_{a,b,c\,(p)} \left(\frac{a^2-4bc}{p}\right) = \frac{p-1}{2}(p^2+p-(p^2-p)) = p(p-1) \tag{4.3}$$

as there are $(p-1)/2$ quadratic residues and non-residues (mod p) each.

This can be used to evaluate more general sums

$$\sum_{a,b,c\,(p)} \left(\frac{a^2-4bc}{p^\ell}\right)$$

for $\ell \in \mathbb{N}$. We note that by the complete multiplicativity of the Legendre symbol the above is

$$\sum_{a,b,c\,(p)} \left(\frac{a^2-4bc}{p}\right)^\ell.$$

We consider different cases depending on the parity of ℓ. If ℓ is even we have, using Lemma 4.4,

$$\sum_{a,b,c\,(p)} \left(\frac{a^2-4bc}{p}\right)^\ell = |\{(a,b,c) \in (\mathbb{Z}/p\mathbb{Z})^3 : a^2-4bc \not\equiv 0\,(p))\}| = p^3-p^2 \tag{4.4}$$

On the other hand, if ℓ is odd

$$\sum_{a,b,c\,(p)} \left(\frac{a^2-4bc}{p}\right)^\ell = \sum_{a,b,c\,(p)} \left(\frac{a^2-4bc}{p}\right) = p(p-1)$$

by (4.3).

We will also need to consider the following sums, for $\alpha, \beta \in \mathbb{N}$ (note that the moduli in both Legendre symbols are the same),

$$\sum_{a,b,c\,(p)} \left(\frac{a^2-4bc}{p}\right)^\alpha \left(\frac{(a+\ell_2)^2-4(b+\ell_1)(c+\ell_3)}{p}\right)^\beta.$$

We again divide into cases. Suppose α, β are both even. Then the sum is simply

$$\left|\{(a,b,c) \in (\mathbb{Z}/p\mathbb{Z})^3 :\, a^2-4bc \not\equiv 0\,(p),\, (a+\ell_2)^2-4(b+\ell_1)(c+\ell_3) \not\equiv 0\,(p)\}\right|.$$

By Lemma 4.4 this quantity is $p^3+O(p^2)$.

Assume than one of α, β (say β) is even and the other one is odd. Then the sum is simply

$$\sum_{a,b,c\,(p)} \left(\frac{a^2-4bc}{p}\right)\left(\frac{(a+\ell_2)^2-4(b+\ell_1)(c+\ell_3)}{p}\right)^2 \ll p^2,$$

where the last step follows from Lemma 4.4 and (4.3).

Suppose finally that both α, β are odd. Then the sum is given by

$$\sum_{a,b,c\,(p)} \left(\frac{a^2-4bc}{p}\right)\left(\frac{(a+\ell_2)^2-4(b+\ell_1)(c+\ell_3)}{p}\right). \tag{4.5}$$

Note that by (4.4) this sum equals $p^2(p-1)$ if $(\ell_1,\ell_2,\ell_3)\equiv(0,0,0)\,(p)$. Suppose that this is not the case. We will estimate the sum differently according to which of the ℓ_i's is not divisible by p. The following lemma will be useful.

Lemma 4.5 (Theorem 2.1.2. in [BEW98])**.** *Let p be an odd prime and a,b,c be integers with $p\nmid a$. Then*

$$\sum_{x\,(p)}\left(\frac{ax^2+bx+c}{p}\right)=\begin{cases}-\left(\frac{a}{p}\right) & \text{if } b^2-4ac\not\equiv 0\,(p),\\ (p-1)\left(\frac{a}{p}\right) & \text{if } b^2-4ac\equiv 0\,(p).\end{cases}$$

First consider the case $\ell_1\not\equiv 0\,(p)$. We bound the sum (4.5) by the triangle inequality as

$$\ll\sum_{a,b\,(p)}\left|\sum_{c\,(p)}\left(\frac{(a^2-4bc)((a+\ell_2)^2-4(b+\ell_1)(c+\ell_3))}{p}\right)\right|.$$

Note that the numerator is of degree 2 as a polynomial in c and that given a and b modulo p there are at most two values of c modulo p for which the argument of the Legendre symbol is divisible by p thanks to the assumption $\ell_1\not\equiv 0\,(p)$. Thus Lemma 4.5 yields the bound $\ll p^2$ for the sum. The case $\ell_3\not\equiv 0\,(p)$ can be dealt with similarly, but putting the b-sum inside the absolute values instead of the c-sum.

Suppose finally that $\ell_2\not\equiv 0\,(p)$ (and we can at the same time assume that $\ell_1,\ell_3\equiv 0\,(p)$). Clearly, as a polynomial of a, $(a^2-4bc)((a+\ell_2)^2-4bc)$ is not an even power of a polynomial with integer coefficients. Thus applying [Sch76, Theorem 2C', p. 43] we have

$$\left|\sum_{a\,(p)}\left(\frac{(a^2-4bc)((a+\ell_2)^2-4bc)}{p}\right)\right|\le 3p^{1/2}.$$

Hence, by the triangle inequality the whole sum

$$\sum_{a,b,c\,(p)}\left(\frac{a^2-4bc}{p}\right)\left(\frac{(a+\ell_2)^2-4bc}{p}\right)$$

is $\ll p^2\cdot p^{1/2}\ll p^{5/2}$ in this case.

So to summarize, we have proved the following.

Lemma 4.6. *Let p be an odd prime and α,β be natural numbers. Then we have*

$$\sum_{a,b,c\,(p)}\left(\frac{a^2-4bc}{p}\right)^{\alpha}=\begin{cases}p^3-p^2 & \text{if } 2|\alpha,\\ p^2-p & \text{if } 2\nmid\alpha,\end{cases}$$

and

$$\sum_{a,b,c\,(p)}\left(\frac{a^2-4bc}{p}\right)^{\alpha}\left(\frac{(a+\ell_2)^2-4(b+\ell_1)(c+\ell_3)}{p}\right)^{\beta}$$

$$=\begin{cases}p^3+O(p^2) & \text{if } 2|\alpha,\beta,\\ O\left(p^2\right) & \text{if } 2|\alpha,2\nmid\beta \text{ or } 2\nmid\alpha,2|\beta,\\ p^3-p^2 & \text{if } 2\nmid\alpha\beta \text{ and } (\ell_1,\ell_2,\ell_3)\equiv(0,0,0)\,(p),\\ O\left(p^{5/2}\right) & \text{if } 2\nmid\alpha\beta \text{ and } (\ell_1,\ell_2,\ell_3)\not\equiv(0,0,0)\,(p).\end{cases}$$

4.1.2. *Applying Poisson summation.*

Proof of Proposition 4.3. Let s and t be odd natural numbers with $s,t \le k^{1/2}$. Let $\Pi_s := \prod_{p\in P_s} p$, $\Pi_t := \prod_{p\in P_t} p$, and $\Pi_{s,t} := \prod_{p\in P_{s,t}} p$. To estimate the sum

$$\sum_{m,n,r} \left(\frac{r^2-4mn}{s}\right)\left(\frac{(r+\ell_2)^2-4(m+\ell_1)(n+\ell_3)}{t}\right) F_1\left(\frac{m}{k}\right) F_2\left(\frac{n}{k}\right) F_3\left(\frac{r}{k}\right) \tag{4.6}$$

we divide the summands m,n,r into congruence classes modulo $\Pi_s\Pi_t\Pi_{s,t}$. Recall that α_p is the integer with $p^{\alpha_p}||s$ and β_p is the integer with $p^{\beta_p}||t$. By the complete multiplicativity of the Legendre symbol and the Chinese remainder theorem, the sum we are interested in takes the form

$$\sum_{\substack{a_1,b_1,c_1 \,(\Pi_s)\\ a_2,b_2,c_2\,(\Pi_t)\\ a_3,b_3,c_3\,(\Pi_{s,t})}} \prod_{p\in P_s}\left(\frac{a_1^2-4b_1c_1}{p}\right)^{\alpha_p} \prod_{p\in P_t}\left(\frac{(a_2+\ell_2)^2-4(b_2+\ell_1)(c_2+\ell_3)}{p}\right)^{\beta_p} \quad \times$$

$$\prod_{p\in P_{s,t}}\left(\frac{a_3^2-4b_3c_3}{p}\right)^{\alpha_p}\left(\frac{(a_3+\ell_2)^2-4(b_3+\ell_1)(c_3+\ell_3)}{p}\right)^{\beta_p} \sum_{\substack{m\equiv\gamma_1\,(\Pi_s\Pi_t\Pi_{s,t})\\ n\equiv\gamma_2\,(\Pi_s\Pi_t\Pi_{s,t})\\ r\equiv\gamma_3\,(\Pi_s\Pi_t\Pi_{s,t})}} F_1\left(\frac{m}{k}\right) F_2\left(\frac{n}{k}\right) F_3\left(\frac{r}{k}\right),$$

where γ_1 is the unique congruence class $\pmod{\Pi_s\Pi_t\Pi_{s,t}}$ that corresponds to $b_1 \pmod{\Pi_s}$, $b_2 \pmod{\Pi_t}$, $b_3 \pmod{\Pi_{s,t}}$ and γ_2,γ_3 are defined analogously. Each of the sums over m,n,r can be evaluated by Lemma 4.2. Since $s,t \le k^{1/3}$, $\Pi_s\Pi_t\Pi_{s,t} \le k^{2/3}$ and we also have that $\widehat{F_j}(\xi) \ll_A |x|^{-A}$ for each $j=1,2,3$. Hence, we see that the inner sum in the preceding equation is

$$\left(\frac{k}{\Pi_s\Pi_t\Pi_{s,t}}\right)^3 \widehat{F_1}(0)\widehat{F_2}(0)\widehat{F_3}(0) + O_{F_i}(k^{-200}).$$

The main term in the preceding equation is independent of $\gamma_1,\gamma_2,\gamma_3$. Hence, combining the previous two estimates and using the Chinese remainder theorem we see that (4.6) equals

$$\prod_{p\in P_s}\left(\sum_{a,b,c\,(p)}\left(\tfrac{a_1^2-4b_1c_1}{p}\right)^{\alpha_p}\right)\prod_{p\in P_{s,t}}\left(\sum_{a,b,c\,(p)}\left(\tfrac{a_3^2-4b_3c_3}{p}\right)^{\alpha_p}\left(\tfrac{(a_3+\ell_2)^2-4(b_3+\ell_1)(c_3+\ell_3)}{p}\right)^{\beta_p}\right)$$
$$\times\prod_{p\in P_t}\left(\sum_{a,b,c\,(p)}\left(\tfrac{(a_2+\ell_2)^2-4(b_2+\ell_1)(c_2+\ell_3)}{p}\right)^{\beta_p}\right)\left(\tfrac{k}{\Pi_s\Pi_t\Pi_{s,t}}\right)^3\widehat{F_1}(0)\widehat{F_2}(0)\widehat{F_3}(0)+O_{F_i}(k^{-100}).$$

Now the local sums can be evaluated by Lemma 4.6 and we get the claimed result. □

4.2. **Bounds for large moments of Dirichlet polynomials.** In this section we will establish upper bounds for moments of Dirichlet polynomials averaged over pairs of certain fundamental discriminants and these bounds will be a main ingredient in the proof of Proposition 4.1. We assume from here on that $(\ell_1,\ell_2,\ell_3) \neq (0,0,0)$ and $F_j : \mathbb{R} \longrightarrow \mathbb{R}_{\ge 0}$. Recall the definition of $f(s,t)$ from (4.2).

To analyze the function $f(s,t)$ further we let $C \ge 1$ be a sufficiently large absolute constant and define the completely multiplicative function u by $u(p) = 1 + C/p$. Also, write $s = s_2^2 s_1$, $t = t_2^2 t_1$, where t_1, s_1 are squarefree and observe that α_p is odd if and only if $p|s_1$, and β_p is odd if and only if $p|t_1$. Let $g = (s_1,t_1)$ and write $s_1 = gs_0$, $t_1 = gt_0$ and note that $(g, s_0t_0) = 1$ since s_1, t_1 are squarefree. We make the following simple observations

- α_p, β_p are both odd if and only if $p|g$,
- α_p is odd and β_p is even (possibly zero) if and only if $p|s_0$,
- α_p is even (possibly zero) and β_p is odd if and only if $p|t_0$.

Recall $l = \prod_{j:\ell_j \neq 0} \ell_j$. Hence, writing $s = s_2^2 g s_0, t = s_2^2 g t_0$ as above, we have for $(st, l) = 1$ that

$$f(s,t) \ll u(s_2)u(t_2)\frac{C^{\Omega(gs_0t_0)}}{s_0t_0\sqrt{g}} \tag{4.7}$$

since C is sufficiently large.

We now use Proposition 4.3 to estimate moments of certain Dirichlet polynomials. Let $\{a(p)\}_p \subset \mathbb{R}$ with $|a(p)| \leq 2p^{1/4-\delta}$ for some fixed $0 < \delta \leq 1/4$ and $a(p) = 0$ if $p|f_1f_2l$. Define

$$\mathcal{V} := \sum_{p \leq x} \frac{a(p)^2}{p} + C_2,$$

where $C_2 > 0$ is a sufficiently large absolute constant (so $\mathcal{V} > 0$, for example).

Lemma 4.7. *Let $\ell \in \mathbb{N}$. Suppose $x^\ell \leq k^{1/3}$. Then*

$$\sideset{}{'}\sum \left(\sum_{p \leq x} \frac{a(p)(\chi_{d_1}(p) + \chi_{d_2}(p))}{\sqrt{p}}\right)^{2\ell} \ll_{F_i} k^3 \frac{(2\ell)!}{2^\ell \ell!}\Big(2\mathcal{V} + O(1)\Big)^\ell.$$

The preceding lemma will be deduced from the following result.

Lemma 4.8. *Let $u, v \in \mathbb{N}$. Suppose $x^{\max(u,v)} \leq k^{1/3}$. Then*

$$\sideset{}{''}\sum \left(\sum_{p \leq x} \frac{a(p)\left(\frac{d_1}{p}\right)}{\sqrt{p}}\right)^u \left(\sum_{p \leq x} \frac{a(p)\left(\frac{d_2}{p}\right)}{\sqrt{p}}\right)^v \ll_{F_i} k^3 \frac{u!}{2^{\lfloor \frac{u}{2} \rfloor}\lfloor \frac{u}{2} \rfloor!} \frac{v!}{2^{\lfloor \frac{v}{2} \rfloor}\lfloor \frac{v}{2} \rfloor!}\Big(\mathcal{V} + O(1)\Big)^{\lfloor \frac{u}{2} \rfloor + \lfloor \frac{v}{2} \rfloor}. \tag{4.8}$$

Using Lemma 4.8 we will quickly deduce Lemma 4.7.

Proof of Lemma 4.7. Recall that $a(p) = 0$ if $p|f_1f_2l$ so that

$$a(p)(\chi_{d_1}(p) + \chi_{d_2}(p)) = a(p)\left(\left(\frac{f_1^2 d_1}{p}\right) + \left(\frac{f_2^2 d_2}{p}\right)\right).$$

Also, recall F_1, F_2, F_3 are nonnegative. Using non-negativity we drop the conditions on the sum over r, m, n that f_1^2 divides $r^2 - 4mn$, f_2^2 divides $(r + \ell_1)^2 - 4(m + \ell_2)(n + \ell_3)$ and both $(r^2 - 4mn)/f_1^2, ((r + \ell_1)^2 - 4(m + \ell_2)(n + \ell_3))/f_2^2$ are fundamental discriminants and use the previous observation to see that

$$\sideset{}{'}\sum \left(\sum_{p \leq x} \frac{a(p)(\chi_{d_1}(p) + \chi_{d_2}(p))}{\sqrt{p}}\right)^{2\ell}$$
$$\leq \sum_{r,m,n} \left(\sum_{p \leq x} \frac{a(p)((\frac{r^2-4mn}{p}) + (\frac{(r+\ell_2)^2-4(m+\ell_1)(n+\ell_3)}{p}))}{\sqrt{p}}\right)^{2\ell} F_1\left(\frac{r}{k}\right)F_2\left(\frac{m}{k}\right)F_3\left(\frac{n}{k}\right).$$

The right-hand side is

$$\sideset{}{''}\sum \left(\sum_{p \leq x} \frac{a(p)((\frac{d_1}{p}) + (\frac{d_2}{p}))}{\sqrt{p}}\right)^{2\ell} \ll_{F_i} k^3 \sum_{j=0}^{2\ell} \binom{2\ell}{j} \frac{j!}{2^{\lfloor \frac{j}{2} \rfloor}\lfloor \frac{j}{2} \rfloor!} \frac{(2\ell - j)!}{2^{\lfloor \frac{2\ell-j}{2} \rfloor}\lfloor \frac{2\ell-j}{2} \rfloor!}\Big(\mathcal{V} + O(1)\Big)^{\lfloor \frac{j}{2} \rfloor + \lfloor \frac{2\ell-j}{2} \rfloor}$$

by Lemma 4.8. The contribution of the even terms to the sum equals

$$\Big(\mathcal{V} + O(1)\Big)^\ell \frac{(2\ell)!}{2^\ell} \sum_{j=0}^{\ell} \frac{1}{j!(\ell - j)!} = \Big(\mathcal{V} + O(1)\Big)^\ell \frac{(2\ell)!}{2^\ell \ell!} \sum_{j=0}^{\ell} \frac{\ell!}{j!(\ell - j)!} = \frac{(2\ell)!}{\ell!}\Big(\mathcal{V} + O(1)\Big)^\ell.$$

The contribution of the odd terms equals

$$\Big(\mathcal{V}+O(1)\Big)^{\ell-1}\frac{(2\ell)!}{2^{\ell-1}}\sum_{j=0}^{\ell-1}\frac{1}{j!((\ell-1)-j)!}=\frac{(2\ell)!}{(\ell-1)!}\Big(\mathcal{V}+O(1)\Big)^{\ell-1}<\frac{(2\ell)!}{\ell!}\Big(\mathcal{V}+O(1)\Big)^{\ell},$$

where in the last step we used the inequality $mw<(1+w)^m$ (which holds for any $m\geq 1$ and $w>0$) with $m=\ell$, $w=1/\mathcal{V}$. □

4.3. **Preliminary estimates.** We will now use Proposition 4.3 and (4.7) to estimate the left-hand side of (4.8). To state the next result, let ν be the multiplicative function with $\nu(p^a)=1/a!$. We note for $m,n\in\mathbb{N}$ that

$$\nu(mn)\leq\nu(m)\nu(n)\qquad\text{and}\qquad\nu(n^2)\leq\frac{\nu(n)}{2^{\Omega(n)}},\tag{4.9}$$

which we will use later. Given $j\in\mathbb{N}$ and a completely multiplicative function b, we see that

$$\Big(\sum_{p\leq x}b(p)\Big)^j=\sum_{n\geq 1}b(n)\sum_{\substack{p_1,\dots,p_j\leq x\\ p_1\cdots p_j=n}}1=j!\sum_{\substack{p|n\Rightarrow p\leq x\\ \Omega(n)=j}}b(n)\nu(n).\tag{4.10}$$

Lemma 4.9. *Let u,v be nonnegative integers. Suppose that $x^{\max(u,v)}\leq k^{1/3}$. Then*

(4.11)

$$\begin{aligned}&\sum\nolimits''\Big(\sum_{p\leq x}\frac{a(p)\left(\frac{d_1}{p}\right)}{\sqrt{p}}\Big)^u\Big(\sum_{p\leq x}\frac{a(p)\left(\frac{d_2}{p}\right)}{\sqrt{p}}\Big)^v\\ &\ll_{F_i}k^3u!v!\sum_{\substack{p|s_2t_2gs_0t_0\Rightarrow p\leq x\\ \Omega(s_2^2gs_0)=u\\ \Omega(t_2^2gt_0)=v}}\frac{a(s_2)^2a(t_2)^2\nu(s_2)\nu(t_2)u(s_2)u(t_2)}{2^{\Omega(s_2t_2)}s_2t_2}\frac{(4C)^{\Omega(gs_0t_0)}\nu(g)\nu(s_0)\nu(t_0)}{(gs_0t_0)^{1+\delta}}+k^{-50}.\end{aligned}$$

Proof. Using (4.10) we get that

$$\Big(\sum_{p\leq x}\frac{a(p)\left(\frac{d}{p}\right)}{\sqrt{p}}\Big)^u=u!\sum_{\substack{p|s\Rightarrow p\leq x\\ \Omega(s)=u}}\frac{a(s)\nu(s)}{\sqrt{s}}\Big(\frac{s}{p}\Big).$$

Applying (4.1) and noting $x^u,x^v\leq k^{1/3}$ we have that the left-hand side of (4.11) is

$$\ll_{F_i}k^3u!v!\sum_{\substack{p|st\Rightarrow p\leq x\\ \Omega(s)=u\\ \Omega(t)=v}}\frac{a(s)a(t)\nu(s)\nu(t)}{\sqrt{st}}f(s,t)+k^{-99}.$$

We write $s=s_2^2gs_0$, $t=t_2^2gt_0$ as in (4.7). Using (4.7), (4.9) and recalling that $|a(p)|\leq 2p^{1/4-\delta}$ yields the claim. □

4.4. **Sum estimates and the proof of Lemma 4.8.** Given a nonnegative integer n we let

$$\eta_n:=\frac{1+(-1)^{n+1}}{2}$$

and note that $\lfloor n/2\rfloor=(n-\eta_n)/2$. We will first establish the following bound.

Lemma 4.10. *Let $j \in \mathbb{N}$. Then*

$$\sum_{\substack{p|s_2s_0\Rightarrow p\le x\\ \Omega(s_2^2s_0)=j}} \frac{a(s_2)^2\nu(s_2)u(s_2)}{2^{\Omega(s_2)}s_2}\frac{(4C)^{\Omega(s_0)}\nu(s_0)}{s_0^{1+\delta}} \ll \frac{(\mathcal{V}+O(1))^{\lfloor\frac{j}{2}\rfloor}}{2^{\lfloor\frac{j}{2}\rfloor}\lfloor\frac{j}{2}\rfloor!}. \tag{4.12}$$

Proof. Write $r=\Omega(s_0)$. The left-hand side of (4.12) is

$$\begin{aligned}&\sum_{\substack{0\le r\le j\\ 2|j-r}}\Bigg(\sum_{\substack{p|s_0\Rightarrow p\le x\\ \Omega(s_0)=r}}\frac{(4C)^{\Omega(s_0)}\nu(s_0)}{s_0^{1+\delta}}\Bigg)\Bigg(\sum_{\substack{p|s_2\Rightarrow p\le x\\ \Omega(s_2)=\frac{j-r}{2}}}\frac{a(s_2)^2\nu(s_2)u(s_2)}{2^{\Omega(s_2)}s_2}\Bigg)\\ &=\sum_{\substack{0\le r\le j\\ 2|j-r}}\frac{1}{r!}\Bigg(\sum_{p\le x}\frac{4C}{p^{1+\delta}}\Bigg)^r\frac{1}{(\frac{j-r}{2})!}\Bigg(\sum_{p\le x}\frac{a(p)^2u(p)}{2p}\Bigg)^{\frac{j-r}{2}},\end{aligned} \tag{4.13}$$

where we have used (4.10) in the last step. Let $C_1=4C\sum_p p^{-1-\delta}$. Also note that

$$\sum_{p\le x}\frac{a(p)^2u(p)}{p}=\mathcal{V}+O(1).$$

Apply the inequality $m^n(m-n)!\ge m!$ with $m=(j-\eta_j)/2$, $n=(r-\eta_j)/2$ to get that

$$\frac{1}{(\frac{j-r}{2})!}\Bigg(\frac{\mathcal{V}}{2}+O(1)\Bigg)^{\frac{j-r}{2}}\le\frac{(\mathcal{V}+O(1))^{\lfloor\frac{j}{2}\rfloor}}{2^{\lfloor\frac{j}{2}\rfloor}\lfloor\frac{j}{2}\rfloor!}\Bigg(\frac{j-\eta_j}{2}\cdot\frac{1}{\mathcal{V}/2+O(1)}\Bigg)^{\frac{r-\eta_j}{2}}. \tag{4.14}$$

Hence, the right-hand side of (4.13) is

$$\begin{aligned}&\le\frac{(\mathcal{V}+O(1))^{\lfloor\frac{j}{2}\rfloor}}{2^{\lfloor\frac{j}{2}\rfloor}\lfloor\frac{j}{2}\rfloor!}\sum_{\substack{0\le r\le j\\ 2|j-r}}\frac{C_1^r}{r!}\Bigg(\frac{j-\eta_j}{\mathcal{V}+O(1)}\Bigg)^{\frac{r-\eta_j}{2}}\le C_1\frac{(\mathcal{V}+O(1))^{\lfloor\frac{j}{2}\rfloor}}{2^{\lfloor\frac{j}{2}\rfloor}\lfloor\frac{j}{2}\rfloor!}\sum_{r=\eta_j}^{\infty}\frac{\Big(C_1\sqrt{\frac{j-\eta_j}{\mathcal{V}+O(1)}}\Big)^{r-\eta_j}}{(r-\eta_j)!}\\ &=C_1\frac{(\mathcal{V}+O(1))^{\lfloor\frac{j}{2}\rfloor}}{2^{\lfloor\frac{j}{2}\rfloor}\lfloor\frac{j}{2}\rfloor!}\exp\Bigg(C_1\sqrt{\frac{j-\eta_j}{\mathcal{V}+O(1)}}\Bigg)\ll\frac{(\mathcal{V}+O(1))^{\lfloor\frac{j}{2}\rfloor}}{2^{\lfloor\frac{j}{2}\rfloor}\lfloor\frac{j}{2}\rfloor!}\exp\Bigg(\frac{C_1\sqrt{2}}{\mathcal{V}+O(1)}\Big\lfloor\frac{j}{2}\Big\rfloor\Bigg),\end{aligned}$$

where in the last step we used that $\exp(\sqrt{x})<3\exp(x)$ for any $x>0$. We complete the proof by noting that

$$\exp\Bigg(\frac{C_1\sqrt{2}}{\mathcal{V}+O(1)}\Big\lfloor\frac{j}{2}\Big\rfloor\Bigg)=\Bigg(1+O\Bigg(\frac{1}{\mathcal{V}}\Bigg)\Bigg)^{\lfloor\frac{j}{2}\rfloor}.$$

□

We are now ready to prove Lemma 4.8.

Proof of Lemma 4.8. Applying Lemma 4.9 it suffices to estimate the sum on the right-hand side of (4.11). Writing $\Omega(g)=r$ we see that the sum on the right-hand side of (4.11) equals

$$\begin{aligned}\sum_{r=0}^{\min\{u,v\}}&\Bigg(\sum_{\substack{p|g\Rightarrow p\le x\\ \Omega(g)=r}}\frac{(4C)^{\Omega(g)}\nu(g)}{g^{1+\delta}}\Bigg)\Bigg(\sum_{\substack{p|s_2s_0\Rightarrow p\le x\\ \Omega(s_2^2s_0)=u-r}}\frac{a(s_2)^2\nu(s_2)u(s_2)}{2^{\Omega(s_2)}s_2}\frac{(4C)^{\Omega(s_0)}\nu(s_0)}{s_0^{1+\delta}}\Bigg)\\ &\times\Bigg(\sum_{\substack{p|t_2t_0\Rightarrow p\le x\\ \Omega(t_2^2t_0)=v-r}}\frac{a(t_2)^2\nu(t_2)u(t_2)}{2^{\Omega(t_2)}t_2}\frac{(4C)^{\Omega(t_0)}\nu(t_0)}{t_0^{1+\delta}}\Bigg).\end{aligned}$$

As before, write $C_1 = 4C\sum_p p^{-1-\delta}$. Using (4.10) to estimate the first inner sum and Lemma 4.10 to bound the second and third inner sums, we see that the above is

$$\ll \sum_{r=0}^{\min\{u,v\}} \frac{C_1^r}{r!} \frac{(\mathcal{V}+O(1))^{\lfloor\frac{u-r}{2}\rfloor+\lfloor\frac{v-r}{2}\rfloor}}{2^{\lfloor\frac{u-r}{2}\rfloor+\lfloor\frac{v-r}{2}\rfloor}\lfloor\frac{u-r}{2}\rfloor!\lfloor\frac{v-r}{2}\rfloor!}. \tag{4.15}$$

Note that $\eta_{u-r} = \eta_u + (-1)^u\eta_r$ so that

$$\Big\lfloor\frac{u-r}{2}\Big\rfloor = \frac{u-r-(\eta_u+(-1)^u\eta_r)}{2} = \Big\lfloor\frac{u}{2}\Big\rfloor - \frac{r+(-1)^u\eta_r}{2}.$$

We now apply the inequality $m^n(m-n)! \geq m!$ twice; first with $m = \lfloor u/2\rfloor$ and $n = (r+(-1)^u\eta_r)/2$, next with $m = \lfloor v/2\rfloor$ and $n = (r+(-1)^v\eta_r)/2$ (cf. (4.14)) to get that (4.15) is

$$\leq \frac{(\mathcal{V}+O(1))^{\lfloor\frac{u}{2}\rfloor+\lfloor\frac{v}{2}\rfloor}}{2^{\lfloor\frac{u}{2}\rfloor+\lfloor\frac{v}{2}\rfloor}\lfloor\frac{u}{2}\rfloor!\lfloor\frac{v}{2}\rfloor!} \sum_{r=0}^{\infty} \frac{C_1^r}{r!}\left(\frac{\lfloor\frac{u}{2}\rfloor}{\frac{1}{2}\mathcal{V}+O(1)}\right)^{\frac{r+(-1)^u\eta_r}{2}}\left(\frac{\lfloor\frac{v}{2}\rfloor}{\frac{1}{2}\mathcal{V}+O(1)}\right)^{\frac{r+(-1)^v\eta_r}{2}}. \tag{4.16}$$

To bound the sum, we apply the Cauchy-Schwarz inequality to see that it is

$$\leq \left(\sum_{r=0}^{\infty}\frac{C_1^r}{r!}\left(\frac{\lfloor\frac{u}{2}\rfloor}{\frac{1}{2}\mathcal{V}+O(1)}\right)^{r+(-1)^u\eta_r}\right)^{1/2}\left(\sum_{r=0}^{\infty}\frac{C_1^r}{r!}\left(\frac{\lfloor\frac{v}{2}\rfloor}{\frac{1}{2}\mathcal{V}+O(1)}\right)^{r+(-1)^v\eta_r}\right)^{1/2}. \tag{4.17}$$

Using that $x \leq e^x$ we have

$$\left(\frac{\lfloor\frac{u}{2}\rfloor}{\frac{\mathcal{V}}{2}+O(1)}\right)^{\eta_r} \leq \exp\left(\frac{\lfloor\frac{u}{2}\rfloor}{\frac{\mathcal{V}}{2}+O(1)}\right)$$

so that

$$\sum_{r=0}^{\infty}\frac{C_1^r}{r!}\left(\frac{\lfloor\frac{u}{2}\rfloor}{\frac{1}{2}\mathcal{V}+O(1)}\right)^{r+\eta_r} \leq \exp\left(\frac{(C_1+1)\lfloor\frac{u}{2}\rfloor}{\frac{1}{2}\mathcal{V}+O(1)}\right).$$

Using these bounds in the right-hand side of (4.17), together with their analogues for the second sum, we get that the right-hand side of (4.16) is

$$\ll \frac{(\mathcal{V}+O(1))^{\lfloor\frac{u}{2}\rfloor+\lfloor\frac{v}{2}\rfloor}}{2^{\lfloor\frac{u}{2}\rfloor+\lfloor\frac{v}{2}\rfloor}\lfloor\frac{u}{2}\rfloor!\lfloor\frac{v}{2}\rfloor!}\exp\left(\frac{4C_1(\lfloor\frac{u}{2}\rfloor+\lfloor\frac{v}{2}\rfloor)}{\mathcal{V}+O(1)}\right) = \frac{(\mathcal{V}+O(1))^{\lfloor\frac{u}{2}\rfloor+\lfloor\frac{v}{2}\rfloor}}{2^{\lfloor\frac{u}{2}\rfloor+\lfloor\frac{v}{2}\rfloor}\lfloor\frac{u}{2}\rfloor!\lfloor\frac{v}{2}\rfloor!},$$

which completes the proof. □

4.5. **Proof of Proposition 4.1.** Having proved Lemma 4.7, we are now in the position to use it to prove Proposition 4.1. We first require a few preliminary lemmas. Recall that $L(s,f) = \prod_p(1-\alpha_p p^{-s})^{-1}(1-\beta_p p^{-s})^{-1}$ for $\mathrm{Re}(s) > 1$, where α_p, β_p are the Satake parameters, so that Deligne's bound gives $|\alpha_p| = |\beta_p| = 1$ for $(p,N) = 1$. We first require the following bound for the central L-values which is due to Chandee.

Lemma 4.11. *Assume GRH for $L(s, f\otimes\chi_d)$. Let d be a fundamental discriminant. Then for $x \geq 2$ there exists $C_0 > 1$ which depends at most on N such that*

$$\log L(\tfrac{1}{2}, f\otimes\chi_d) \leq \sum_{\substack{p^n\leq x\\ p\nmid N}} \frac{(\alpha_p^n+\beta_p^n)\chi_d(p)^n}{np^{\frac{n}{2}(1+2/\log x)}}\frac{\log x/p^n}{\log x} + C_0\frac{\log|dk|}{\log x}.$$

Remark 3. *In particular, choosing $x = \log|dk|$ and using Deligne's bound $|\alpha_p| = |\beta_p| = 1$ for $p\nmid N$ we have that*

$$L(\tfrac{1}{2}, f\otimes\chi_d) \ll_N \exp\left(\frac{2C_0\log|dk|}{\log\log(|dk|+1)}\right), \tag{4.18}$$

which we will use later.

Proof. See Theorem 2.1 of Chandee [Cha09]. □

We next record the following estimate, which follows from a classical argument of Littlewood (see Titchmarsh [Tit86, Eq. (14.2.2)] or [LR20, Lemma 5.3]). Assuming GRH for $L(s, \mathrm{sym}^2 f)$ we have for $x \geq 2$ that

$$\sum_{p \leq x} \frac{\lambda_f(p)^2}{p} = \log\log x + O(\log\log\log k). \tag{4.19}$$

Finally, we require the following estimate for large deviations of Dirichlet polynomials. For $x \geq 2$ and d_1, d_2 fundamental discriminants, let

$$P(d_1, d_2; x) := \sum_{\substack{p \leq x \\ p \nmid f_1 f_2 l N}} \frac{\lambda_f(p)(\chi_{d_1}(p) + \chi_{d_2}(p))}{p^{1/2+1/\log x}} \frac{\log x/p}{\log x}$$

(recall $l = \prod_{j:\ell_j \neq 0} \ell_j$). Also, for $V \in \mathbb{R}$, $x \geq 2$ let

$$A_k(V; x) := \sum\nolimits^{'} 1_{(V,\infty)}(P(d_1, d_2; x)).$$

Lemma 4.12. *Let $\varepsilon > 0$ be sufficiently small. Suppose that $V \geq (\log\log k)^{3/4}$. Then we have that*

$$A_k(V; k^{1/(\varepsilon V)}) \ll k^3 \exp\left(\frac{-V^2(1-2\varepsilon)}{4\log\log k}\right) + k^3\, e^{-\frac{\varepsilon}{4} V \log V}.$$

Proof. Define $x = k^{1/(\varepsilon V)}$, $z = x^{1/\log\log k}$, $V_1 = (1 - \frac{\varepsilon}{2})V$ and $V_2 = \frac{\varepsilon}{2}V$. Also let $Q(d_1, d_2; x) = P(d_1, d_2; x) - P(d_1, d_2; z)$. Clearly, if $P(d_1, d_2; x) > V$ then $P(d_1, d_2; z) > V_1$ or $Q(d_1, d_2; x) > V_2$. We first bound the frequency with which the former occurs using Markov's inequality and Lemma 4.7 to get for $\ell \leq \frac{\varepsilon}{3} V \log\log k$ that

$$\begin{aligned}\sum\nolimits^{'} 1_{(V_1,\infty)}(P(d_1, d_2; z)) \leq & \frac{1}{V_1^{2\ell}} \sum\nolimits^{'} P(d_1, d_2; z)^{2\ell} \\ \ll & k^3 \frac{(2\ell)!}{(V_1^2 2)^\ell \ell!} \left(2 \sum_{p \leq k} \frac{\lambda_f(p)^2}{p} + O(1)\right)^\ell,\end{aligned}$$

where we have also extended the inner sum on the right-hand side using nonnegativity. Applying Stirling's formula together with (4.19) the right-hand side above is

$$\ll k^3 \left(\frac{4\ell \log\log k}{V_1^2 e}(1 + \varepsilon^3)\right)^\ell.$$

In the range $V \leq \varepsilon(\log\log k)^2$ we choose $\ell = \lfloor V^2/(4\log\log k)\rfloor$, whereas for larger V we take $\ell = \lfloor \varepsilon V/3 \rfloor$. This gives that

$$\sum\nolimits^{'} 1_{(V_1,\infty)}(P(d_1, d_2; z)) \ll k^3 \exp\left(\frac{-V^2(1-2\varepsilon)}{4\log\log k}\right) + k^3\, e^{-\frac{\varepsilon}{7} V \log V}.$$

To bound how often $Q(d_1, d_2; x) > V_2$ we argue similarly and note that

$$\sum_{z < p \leq x} \frac{\lambda_f(p)^2}{p} \leq 4 \log \frac{\log x}{\log z} + o(1).$$

to see that for $\ell = \lfloor \frac{\varepsilon}{3} V \rfloor$

$$\sum\nolimits^{'} 1_{(V_2,\infty)}(Q(d_1, d_2; z)) \ll k^3 \frac{(2\ell)!}{(V_2^2 2)^\ell \ell!} \left(8 \log\log\log k(1 + o(1))\right)^\ell \ll k^3\, e^{\frac{-\varepsilon}{7} V \log V},$$

for $V \geq (\log\log k)^{3/4}$. $\square$

Proof of Proposition 4.1. We first will record a bound for $L(\frac{1}{2}, f \otimes \chi_d)$. In Lemma 4.11, bounding the contribution from the prime powers with $n \geq 3$ trivially we have that

$$\begin{aligned}\log L(\tfrac{1}{2}, f \otimes \chi_d) \leq \sum_{\substack{p \leq x \\ p \nmid lNf_1f_2}} \frac{\lambda_f(p)\chi_d(p)}{p^{1/2+1/\log x}} \frac{\log x/p}{\log x} + \frac{1}{2}\sum_{\substack{p \leq x \\ p \nmid N}} \frac{(\alpha_p^2+\beta_p^2)\chi_d(p)^2}{p^{1+2/\log x}} \frac{\log x/p^2}{\log x} \\ + 2\sum_{p \mid lf_1f_2} \frac{1}{\sqrt{p}} + \frac{C_0 \log|dk|}{\log x} + O(1),\end{aligned} \tag{4.20}$$

where we have used that $\lambda_f(p) = \alpha_p + \beta_p$. Also, $\alpha_p^2 + \beta_p^2 = \lambda_f(p)^2 - 2$ using this together with (4.19) we get that the second term on the right-hand side above is, for $|d| \leq k^3$,

$$\begin{aligned}&\leq \frac{1}{2}\sum_{p \leq k} \frac{\lambda_f(p)^2}{p} - \log\log k + \sum_{x < p \leq k} \frac{1}{p} + \sum_{p \mid d} \frac{1}{p} + O(1) \\ &\leq -\frac{1}{2}\log\log k + \frac{\log k}{\log x} + O(\log\log\log k),\end{aligned} \tag{4.21}$$

where in the previous estimate we also used the inequality $\log t \leq t$, for $t > 0$.

Let $L(d_1, d_2) := L(\frac{1}{2}, f \otimes \chi_{d_1})L(\frac{1}{2}, f \otimes \chi_{d_2}) \exp(-4\sum_{p \mid lf_1f_2} \frac{1}{\sqrt{p}})$. Also, let

$$B_k(V) := \sum\nolimits' 1_{(e^V, \infty)}(L(d_1, d_2)).$$

Observe that we have

$$\begin{aligned}\sum\nolimits' \sqrt{L(d_1, d_2)} &= \frac{1}{2}\int_{\mathbb{R}} e^{V/2} B_k(V)\, \mathrm{d}V \\ &= \frac{1}{2(\log k)^{1/2}} \int_{\mathbb{R}} e^{V/2} B_k(V - \log\log k)\, \mathrm{d}V.\end{aligned} \tag{4.22}$$

Since the contribution from $V \leq 2(\log\log k)^{3/4}$ is $O(k^3(\log k)^{-1/2+o(1)})$ and by (4.18) $B_k(V) = O(k^{-100})$ for $V \geq 16C_0 \log k/\log\log k$ (here we also used that F_1, F_2, F_3 decay rapidly), it suffices to restrict to V in the remaining range. Using (4.20) and (4.21) we see that for $x \geq 2$ and $|d_1|, |d_2| \leq k^3$ that

$$\log L(d_1, d_2) \leq P(d_1, d_2; x) - \log\log k + 18C_0 \frac{\log k}{\log x} + O(\log\log\log k),$$

so that choosing $x = k^{1/(\varepsilon V)}$ we have that $B_k(V - \log\log k) \leq A_k(V(1 - 19C_0\varepsilon); k^{1/(\varepsilon V)})$. Using this inequality together with the identity

$$\int_{\mathbb{R}} e^{-\frac{t^2}{4\log\log k} + \frac{t}{2}}\, \mathrm{d}t = 2\sqrt{\pi \log\log k}\, (\log k)^{1/4}$$

and applying Lemma 4.12 we have that the right-hand side of (4.22) is

$$\ll \frac{k^3}{(\log k)^{1/2}} \int_{2(\log\log k)^{3/4}}^{17C_0 \frac{\log k}{\log\log k}} e^{V/2}\left(e^{-\frac{V^2(1-39C_0\varepsilon)}{4\log\log k}} + e^{-\frac{\varepsilon}{7}V\log V}\right)\mathrm{d}V + k^3(\log k)^{-1/2+o(1)} \ll \frac{k^3}{(\log k)^{1/4-\varepsilon}},$$

which completes the proof. $\square$

4.6. **Shifted convolution sum for the off-diagonal.** Finally we move to deduce Proposition 3.1 from Proposition 4.1. Recall that $F \in S_k(\Gamma)$ is a *Saito–Kurokawa lift* and a Hecke eigenform, and we let $R(T)$ denote the normalized Fourier coefficients of F. Let $\widetilde{f} \in S_{k-\frac{1}{2}}(\Gamma_0(4))$ be the classical half-integral weight form that F is lifted from [EZ85, §6], and let $c(n)$ denote its normalized Fourier coefficients. We let $f \in S_{2k-2}(\mathrm{SL}_2(\mathbb{Z}))$ be the normalized Hecke eigenform associated to $\widetilde{f}$ via the Shimura correspondence.

We first record the following identities. For a negative integer ℓ, let $w(\ell) = 4$ if $\ell = -4$, $w(\ell) = 6$ if $\ell = -3$ and $w(\ell) = 2$ otherwise. Given a fundamental discriminant $d < 0$ and $a \in \mathbb{N}$ we have

$$h(a^2 d) = \frac{w(a^2 d)}{w(d)} h(d) a \sum_{t|a} \frac{\mu(t)\chi_d(t)}{t} = \frac{a\sqrt{|d|} w(a^2 d)}{2\pi} L(1, \chi_d) \sum_{t|a} \frac{\mu(t)\chi_d(t)}{t}, \tag{4.23}$$

see [Coh93, Remark, p. 233 & Proposition 5.3.12]. Additionally,

$$c(a^2|d|) = c(|d|) \sum_{uv=a} \frac{\mu(u)\chi_d(u)}{\sqrt{u}} \lambda_f(v). \tag{4.24}$$

Recall that

$$c_k = \frac{\Gamma(k)\Gamma\left(k - \frac{1}{2}\right)}{3 \cdot 2^{2k+1} \cdot \pi^{2k+\frac{1}{2}}}.$$

Recall that $\mathcal{D}$ denotes the set of negative fundamental discriminants.

Lemma 4.13. *Let $d \in \mathcal{D}$ and $h \in \mathbb{N}$. Then*

$$\frac{c_k}{\|F\|_2^2} \sum_{\substack{T \in \Lambda_2^+ / \mathrm{SL}_2(\mathbb{Z}) \\ \mathrm{disc}(T) = h^2 d}} |R(T)|^2 \ll_\varepsilon h^{1+\varepsilon} \sqrt{|d|} \frac{L(1, \chi_d) L(\frac{1}{2}, f \otimes \chi_d)}{L(1, \mathrm{sym}^2 f)}. \tag{4.25}$$

Proof. Write $\tau(n) := \sum_{d|n} 1$. Given $T \in \Lambda_2^+/\mathrm{SL}_2(\mathbb{Z})$ with $\mathrm{disc}(T) = h^2 d$ write $\mathrm{cont}(T) = g$. Recalling (3.3) and applying Cauchy-Schwarz we have that

$$|R(T)|^2 \leq \tau(g) \sum_{j|g} j \left| c\left(\tfrac{h^2|d|}{j^2}\right) \right|^2.$$

Using (4.24) and the bound $\lambda_f(n) \ll_\varepsilon n^\varepsilon$ we have that $c(\frac{h^2|d|}{j^2}) \ll h^\varepsilon |c(|d|)|$. We will next apply (3.5) and note that in the notation of Section 3.2, the automorphic representation π_0 is generated by f and so $L(s, \pi_0) = L(s, f)$, $L(s, \pi_0 \otimes \chi_d) = L(s, f \otimes \chi_d)$ and $L(s, \mathrm{sym}^2 \pi_0) = L(s, \mathrm{sym}^2 f)$. Hence since $g | 2h$, applying (3.5) and (3.4) we get that

$$\frac{c_k}{\|F\|_2^2} |R(T)|^2 \ll \frac{c_k}{\|F\|_2^2} \sum_{j|g} j \left| c\left(\tfrac{h^2|d|}{j^2}\right) \right|^2 \ll_\varepsilon h^\varepsilon \frac{L(\frac{1}{2}, f \otimes \chi_d)}{L(1, \mathrm{sym}^2 f)} \sum_{j|g} j. \tag{4.26}$$

This gives that

$$\frac{c_k}{\|F\|_2^2} \sum_{\substack{T \in \Lambda_2^+ / \mathrm{SL}_2(\mathbb{Z}) \\ \mathrm{disc}(T) = h^2 d}} |R(T)|^2 \ll_\varepsilon h^\varepsilon \frac{L(\frac{1}{2}, f \otimes \chi_d)}{L(1, \mathrm{sym}^2 f)} \sum_{g|2h} \sum_{j|g} j \sum_{\substack{T \in \Lambda_2^+ / \mathrm{SL}_2(\mathbb{Z}) \\ \mathrm{disc}(T) = h^2 d \\ \mathrm{cont}(T) = g}} 1. \tag{4.27}$$

In the sum over T above we pass to counting *primitive* T, and write $T = gT'$ where $\mathrm{cont}(T') = 1$ and apply (4.23) to get that

$$\sum_{\substack{T\in\Lambda_2^+/\mathrm{SL}_2(\mathbb{Z})\\ \mathrm{disc}(T)=h^2d\\ \mathrm{cont}(T)=g}} 1 = \sum_{\substack{T'\in\Lambda_2^+/\mathrm{SL}_2(\mathbb{Z})\\ \mathrm{disc}(T')=h^2d/g^2\\ \mathrm{cont}(T')=1}} 1 \ll_\varepsilon \frac{h^{1+\varepsilon}}{g}\sqrt{|d|}L(1,\chi_d).$$

Applying the preceding bound in (4.27) completes the proof. □

Lemma 4.14. *Assume GRH. Let $\varepsilon > 0$. For $1 \le H \le k^{1/3}$ we have that*

$$\begin{aligned}
&\frac{c_k}{\|F\|_2^2}\sum_{T\in\Lambda_2^+} |R(T)R(T+L)|\, F_1\Big(\frac{m}{k}\Big)F_2\Big(\frac{n}{k}\Big)F_3\Big(\frac{r}{k}\Big)\\
&= \frac{c_k}{\|F\|_2^2}\sum_{h_1,h_2\le H}\sum_{\substack{r,m,n\in\mathbb{Z}\\ (r^2-4mn)/h_1^2\in\mathcal{D}\\ ((r+\ell_2)^2-4(m+\ell_1)(n+\ell_3))/h_2^2\in\mathcal{D}}} |R(T)R(T+L)|\, F_1\Big(\frac{m}{k}\Big)F_2\Big(\frac{n}{k}\Big)F_3\Big(\frac{r}{k}\Big)\\
&\quad + O_{F_i,\varepsilon}\Big(k^3\frac{(\log k)^\varepsilon}{H^{1/2-\varepsilon}}\Big).
\end{aligned}$$

In the proof we will use Propositions 5.2 and 5.1, which are established in Section 5. Under GLH the former gives an asymptotic for the sum of $|R(T)|^2$ over all $T \in \Lambda_2^+/\mathrm{SL}_2(\mathbb{Z})$ with $\mathrm{disc}(T) \le X$ provided $X \ge k^{1+\varepsilon}$. The latter gives an asymptotic for the sum of $L(1/2, f\otimes\chi_d)$ for $d \le X$ with $X \ge k^{1+\varepsilon}$. The argument below only requires nearly sharp upper bounds.

Proof. We write

$$\begin{aligned}
&\frac{c_k}{\|F\|_2^2}\sum_{T\in\Lambda_2^+} |R(T)R(T+L)|F_1\Big(\frac{m}{k}\Big)F_2\Big(\frac{n}{k}\Big)F_3\Big(\frac{r}{k}\Big)\\
&\qquad = \frac{c_k}{\|F\|_2^2}\sum_{h_1,h_2\in\mathbb{N}}\sum_{d_1,d_2\in\mathcal{D}}\sum_{\substack{T\in\Lambda_2^+\\ \mathrm{disc}(T)=h_1^2d_1\\ \mathrm{disc}(T+L)=h_2^2d_2}} |R(T)R(T+L)|\, F_1\Big(\frac{m}{k}\Big)F_2\Big(\frac{n}{k}\Big)F_3\Big(\frac{r}{k}\Big).
\end{aligned} \tag{4.28}$$

We will bound the contribution of the terms with $h_1 \ge H$ to the sum above. By a similar argument, the terms with $h_2 \ge H$ can be shown to satisfy the same bound. By the Cauchy-Schwarz inequality this part of the sum above is

$$\begin{aligned}
&\le \Bigg(\frac{c_k}{\|F\|_2^2}\sum_{\substack{h_1,h_2\\ h_1\ge H}}\sum_{d_1,d_2\in\mathcal{D}}\sum_{\substack{T\in\Lambda_2^+\\ \mathrm{disc}(T)=h_1^2d_1\\ \mathrm{disc}(T+L)=h_2^2d_2}} |R(T)|^2F_1\Big(\frac{m}{k}\Big)F_2\Big(\frac{n}{k}\Big)F_3\Big(\frac{r}{k}\Big)\Bigg)^{1/2}\\
&\qquad\times\Bigg(\frac{c_k}{\|F\|_2^2}\sum_{T\in\Lambda_2^+}|R(T+L)|^2F_1\Big(\frac{m}{k}\Big)F_2\Big(\frac{n}{k}\Big)F_3\Big(\frac{r}{k}\Big)\Bigg)^{1/2}.
\end{aligned} \tag{4.29}$$

Note that there exists $G \in C_c^\infty(\mathbb{R}_{>0})$ so that for $m, n, r \in \mathbb{Z}$,

$$F_1(m/k)F_2(n/k)F_3(r/k) \ll_L \min(G(|r^2-4mn|/k^2), G(|(r+\ell_3)^2-(m+\ell_1)(n+\ell_2)|/k^2)).$$

Note also that for each $T = \begin{pmatrix} m & r/2 \\ r/2 & n \end{pmatrix}$, there are $\ll 1$ matrices $A \in \mathrm{SL}_2(\mathbb{Z})$ such that ${}^tATA = \begin{pmatrix} m' & r'/2 \\ r'/2 & n' \end{pmatrix}$ satisfies $F_1(m'/k)F_2(n'/k)F_3(r'/k) \neq 0$. Hence, using Proposition 5.2 along with (5.32) and noting $L(1, \mathrm{sym}^2 f) \gg_\varepsilon k^{-\varepsilon}$, the second term in the preceding display is

$$\ll \left(\frac{c_k}{\|F\|_2^2} \sum_{T \in \Lambda_2^+/\mathrm{SL}_2(\mathbb{Z})} \frac{|R(T)|^2}{\varepsilon(T)} G\left(\frac{|\mathrm{disc}(T)|}{k^2} \right) \right)^{1/2} \ll_\varepsilon k^{3/2+\varepsilon}. \tag{4.30}$$

It remains to bound the first factor in (4.29). Note that given L and $T \in \Lambda_2^+/\mathrm{SL}_2(\mathbb{Z})$ the number of $(h_2, d_2) \in \mathbb{N} \times \mathcal{D}$ such that $\mathrm{disc}(T + L) = h_2^2 d_2$ is bounded uniformly with respect to T. Using Lemma 4.13 and recalling $F_1(m/k)F_2(n/k)F_3(r/k) \ll G(h^2 d/k^2)$ we see that the first factor in (4.29) is

$$\ll_\varepsilon k^{1/2} c_k^{1/2} \left(\sum_{h \geq H} h^\varepsilon \sum_{d \in \mathcal{D}} \frac{L(\frac{1}{2}, f \otimes \chi_d) L(1, \chi_d)}{L(1, \mathrm{sym}^2 f)} \left(\frac{h\sqrt{|d|}}{k} G\left(\frac{h^2|d|}{k^2} \right) \right) \right)^{1/2}. \tag{4.31}$$

Under GRH, it is well-known that $L(1, \chi_d) \ll \log\log(|d|+4)$. Additionally, by Proposition 5.1, say, we have for $h \leq k^{1/3}$ under GLH that

$$\sum_{d \in \mathcal{D}} L(\tfrac{1}{2}, f \otimes \chi_d) \left(\frac{h\sqrt{|d|}}{k} G\left(\frac{h^2|d|}{k^2} \right) \right) \asymp \frac{k^2}{h^2} L(1, \mathrm{sym}^2 f). \tag{4.32}$$

In (4.31), we split the range of the sum over h into two ranges $H \leq h \leq k^{1/3}$ and $h > k^{1/3}$ in the latter range we can use GLH to bound the inner sum over $\mathcal{D}$ whereas in the former range of h we use (4.32), applying the resulting bound together with (4.30) in (4.29) completes the proof. □

Now we are finally ready to prove Proposition 3.1.

Proof of Proposition 3.1. Using Lemma 4.14 and (4.26) we have that

$$\frac{c_k}{\|F\|_2^2} \sum_{T \in \Lambda_2^+} |R(T)R(T+L)| \, F_1\left(\frac{m}{k}\right) F_2\left(\frac{n}{k}\right) F_3\left(\frac{r}{k}\right) \ll_{F_i, L, N, \varepsilon} k^3 \frac{(\log k)^\varepsilon}{H^{1/2-\varepsilon}} + \sum_{h_1, h_2 \leq H} (h_1 h_2)^{1/2+\varepsilon}$$

$$\sum_{\substack{r,m,n \\ (r^2-4mn)/h_1^2 = d_1 \in \mathcal{D} \\ ((r+\ell_1)^2 - 4(m+\ell_2)(n+\ell_3))/h_2^2 = d_2 \in \mathcal{D}}} \frac{\sqrt{L(\frac{1}{2}, f \otimes \chi_{d_1}) L(\frac{1}{2}, f \otimes \chi_{d_2})}}{L(1, \mathrm{sym}^2 f) L(\frac{3}{2}, f)} F_1\left(\frac{m}{k}\right) F_2\left(\frac{n}{k}\right) F_3\left(\frac{r}{k}\right).$$

Using Proposition 4.1 with $f_1 = h_1, f_2 = h_2$ and the fact that $L(1, \mathrm{sym}^2 f) \gg_\varepsilon (\log k)^{-\varepsilon}$ under GRH ([Xia16, Theorem 1]), we see that the second term on the right-hand side is

$$\ll_{F_i, L, N, \varepsilon} H^{3+\varepsilon} \frac{k^3}{(\log k)^{1/4-\varepsilon}}.$$

Taking $H = (\log k)^{1/14}$ balances the error terms and completes the proof. □

5. Proof of Proposition 3.2

In this Section, we prove Proposition 3.2. As a starting point, we carry out an asymptotic evaluation of a twisted first moment of central L-values in Section 5.1. By combining the resulting formula with computations involving the Rankin–Selberg convolution of the Koecher–Maass series in Section 5.2, we obtain the proof of Proposition 3.2 when $g = 1$. The proof for the case when g is a cusp form

or a unitary Eisenstein series requires us to reframe the weight function in terms of a toric period and then use Waldspurger's period formula and subconvex bounds for twisted L-functions; this is done in Sections 5.3 and 5.4. Finally in Section 5.5 we complete the proof of Proposition 3.2 by combining the above results with the spectral decomposition of a general $g \in C_c^\infty(\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H})$.

5.1. **A twisted first moment asymptotic.** Let f be a newform of weight $2k-2$ and level 1, we do not need to assume k is even for the next result. Let $\mathcal{R} = \{1, 5, 8, 9, 12, 13\}$ be the set of admissible residue classes for fundamental discriminants modulo 16 and $\eta \in \mathcal{R}$. Also let $\eta_1 = (-1)^{k-1}\eta$ and

$$\mathcal{D}_\eta := \left\{ d = (-1)^{k-1}n : n > 0, \mu^2\left(\frac{n}{(4,n)}\right) = 1,\ n \equiv \eta_1 \,(16) \right\}$$

and

$$L_{f,\eta}(s) := \left(1 - \frac{\lambda_f(2)(\frac{\eta}{2})}{2^s} + \frac{1}{4^s}\right)^{-1},$$

where $(\frac{\eta}{2})$ is the Kronecker symbol. We will use the convention that $(\frac{0}{2})^0 = 1$. The moment result we need is the following.

Proposition 5.1. *Assume GLH. Let $\varepsilon > 0$. Let $\phi \in C_c^\infty(\mathbb{R}_+)$ and $u \in \mathbb{N}$. Write $u = 2^a u_2^2 u_1$ where $2^a || u$ and u_1 is squarefree. Then for $\eta \in \mathcal{R}$ we have that*

$$\begin{aligned}\sum_{d\in\mathcal{D}_\eta} L(\tfrac{1}{2}, f\otimes\chi_d)\chi_d(u)\phi\left(\frac{|d|}{D}\right) &= \left(\frac{\eta_1}{2}\right)^a \frac{D\lambda_f(u_1)}{8\sqrt{u_1}}\left(\int_0^\infty \phi(\xi)\,\mathrm{d}\xi\right) L_{f,\eta}(\tfrac{1}{2}) L(1, \mathrm{sym}^2 f)\mathcal{G}(1;u) \\ &\quad + O_{\phi,\varepsilon}\left((ukD)^\varepsilon \sqrt{u}\, D^{3/4} k^{1/4}\right),\end{aligned} \tag{5.1}$$

where $\mathcal{G}(1;\cdot)$ is a multiplicative function satisfying $\mathcal{G}(1;p^k) = 1 + O(1/p)$ at prime powers.

An explicit expression for $\mathcal{G}(1;\cdot)$ is given in [JLS23, (40)].

Proof. This result follows from an adaptation of the methods developed in [Sou00, SY10, RlS15], however there are a few key differences so we will give a detailed sketch. We assume GLH to streamline the proof and so that our argument more closely mirrors [RlS15]. To detect fundamental discriminants we use the following identity, for $d \in \mathcal{D}_\eta$

$$\sum_{\substack{\alpha^2 | d \\ (\alpha,2)=1}} \mu(\alpha) = \begin{cases} 1 & \text{if } \frac{d}{(4,d)} \text{ is squarefree} \\ 0 & \text{otherwise} \end{cases} \tag{5.2}$$

since $d \equiv \eta\,(16)$ and $\eta \in \mathcal{R}$ implies that $4 \nmid \frac{d}{(4,d)}$. Applying (5.2) and the approximate functional equation for $L(\frac{1}{2}, f\otimes\chi_d)$ we have that

$$\sum_{d\in\mathcal{D}_\eta} L(\tfrac{1}{2}, f\otimes\chi_d)\chi_d(u)\phi\left(\frac{|d|}{D}\right) = 2\sum_{\substack{n\geq 1 \\ (n,2)=1}} \frac{\lambda_f(n)}{\sqrt{n}} \sum_{d\in\mathcal{D}_\eta} \sum_{\substack{\alpha^2|d \\ (\alpha,2)=1}} \mu(\alpha)\chi_d(nu) V\left(\frac{n}{k|d|}\right)\phi\left(\frac{|d|}{D}\right), \tag{5.3}$$

where for $\xi, c > 0$,

$$V(\xi) := \frac{1}{2\pi i}\int_{(c)} \frac{\Gamma(w+k)}{\Gamma(k)k^w}(2\pi\xi)^{-w} L_{f,\eta}(\tfrac{1}{2}+w) e^{w^2}\frac{dw}{w}. \tag{5.4}$$

We first split the sum over α in (5.3) into two ranges $\alpha > \sqrt{Y}$ and $\alpha \leq \sqrt{Y}$.

Following the argument given in [RlS15, §10.1], except that we use GLH in place of the quadratic large sieve[6] to bound $L(\frac{1}{2}, f\otimes\chi_d)$, we get that

$$\Big|\sum_{\substack{n\geq 1\\(n,2)=1}}\frac{\lambda_f(n)}{\sqrt{n}}\sum_{d\in\mathcal{D}_\eta}\sum_{\substack{\alpha^2|d\\(\alpha,2)=1\\\alpha>\sqrt{Y}}}\mu(\alpha)\chi_d(nu)V\Big(\frac{n}{k|d|}\Big)\Big|\ll_\varepsilon (kD)^\varepsilon\frac{D}{\sqrt{Y}}. \tag{5.5}$$

It remains to estimate the terms with $\alpha\leq\sqrt{Y}$. Write $D=k^\delta$. Given $(x,y,z)\in\mathbb{R}^3_{>0}$ let

$$F(x;y,z):=\phi\Big(\frac{x}{y}\Big)V\Big(\frac{z}{xy^{1/\delta}}\Big),\qquad \widehat{F}(\lambda;y,z)=\int_{\mathbb{R}}F(t;y,z)e(-\lambda t)\,dt.$$

Shifting contours in (5.4) we see that $V(\xi)=O_A(|\xi|^{-A})$ and conclude $\widehat{F}(\lambda;y,z)\ll_{A,\phi}y(y^{1+1/\delta}/z)^A$. Also, we can repeatedly integrate by parts to get $\widehat{F}(\lambda;y,z)\ll_{A,\phi}y^{1+A/\delta}(|\lambda|z)^{-A}$. Combining these estimates we have for any $A>0$ that

$$\widehat{F}(\lambda;y,z)\ll_{\phi,A}y\min\bigg\{\Big(\frac{y^{1+1/\delta}}{z}\Big)^A,\Big(\frac{y^{1/\delta}}{z|\lambda|}\Big)^A\bigg\}. \tag{5.6}$$

Write $u=v2^a$ where $2^a||u$. Applying Lemma 4.2 the terms in (5.3) with $\alpha\leq\sqrt{Y}$ are

$$\begin{aligned}&2\Big(\frac{\eta_1}{2}\Big)^a\sum_{\substack{n\geq1\\(n,2)=1}}\frac{\lambda_f(n)}{\sqrt{n}}\sum_{\substack{\alpha\leq\sqrt{Y}\\(\alpha,2)=1}}\mu(\alpha)\Big(\frac{\alpha^2}{nv}\Big)\sum_{r\equiv\eta_1\,(16)}\Big(\frac{r}{nv}\Big)V\Big(\frac{n}{k|r|\alpha^2}\Big)\phi\Big(\frac{r\alpha^2}{D}\Big)\\&=\frac{(\frac{\eta_1}{2})^a}{8\sqrt{v}}\sum_{\substack{\alpha\leq\sqrt{Y}\\(\alpha,2)=1}}\mu(\alpha)\sum_{\substack{n\geq1\\(n,2)=1}}\frac{\lambda_f(n)}{n}\Big(\frac{\alpha^2}{nv}\Big)\sum_{j\in\mathbb{Z}}\frac{\tau_j(nv)}{\sqrt{nv}}e\Big(\frac{j\eta_1\overline{nv}}{16}\Big)\widehat{F}\Big(\frac{j}{16nv};\frac{D}{\alpha^2},\frac{n}{\alpha^{2+2/\delta}}\Big).\end{aligned} \tag{5.7}$$

The contribution from the term with $j=0$ yields the main term in Proposition 5.1. Since $\tau_0(nv)=\varphi(nv)$ if nv is a square and $\tau_0(nv)=0$ otherwise we get the term with $j=0$ in (5.7) equals

$$\frac{(\frac{\eta_1}{2})^a}{8}\sum_{\substack{\alpha\leq\sqrt{Y}\\(\alpha,2v)=1}}\mu(\alpha)\sum_{\substack{n\geq1\\(n,2\alpha)=1\\nv=\square}}\frac{\lambda_f(n)}{\sqrt{n}}\frac{\varphi(nv)}{nv}\widehat{F}\Big(0;\frac{D}{\alpha^2},\frac{n}{\alpha^{2+2/\delta}}\Big). \tag{5.8}$$

Applying the first bound in (5.6), we add back in the terms with $\alpha>\sqrt{Y}$ at the cost of an error term of size $\ll_{\varepsilon,\phi}(uDk)^\varepsilon D/\sqrt{Y}$. Using this estimate and also (5.4) we conclude that the expression in (5.8) equals

$$\begin{aligned}&\frac{(\frac{\eta_1}{2})^aD}{8}\frac{1}{2\pi i}\int_{(c)}\bigg(\int_{\mathbb{R}}\phi(\xi)\xi^s\,\mathrm{d}\xi\bigg)\frac{\Gamma(s+k)}{\Gamma(k)}D^sL_{f,\eta}(s+\tfrac{1}{2})\\&\qquad\times\sum_{(\alpha,2u)=1}\frac{\mu(\alpha)}{\alpha^2}\sum_{\substack{(n,2\alpha)=1\\nv=\square}}\frac{\lambda_f(n)}{n^{s+\frac{1}{2}}}\frac{\varphi(nv)}{nv}e^{s^2}\frac{ds}{s}+O_{\phi,\varepsilon}\Big((kD)^\varepsilon\frac{D}{\sqrt{Y}}\Big).\end{aligned} \tag{5.9}$$

In [JLS23, Eq'ns (40)-(41)] it is shown that

$$\sum_{(\alpha,2v)=1}\frac{\mu(\alpha)}{\alpha^2}\sum_{\substack{(n,2\alpha)=1\\nv=\square}}\frac{\lambda_f(n)}{n^{s+\frac{1}{2}}}\frac{\varphi(nv)}{nv}=\frac{\lambda_f(u_1)}{u_1^{s+\frac{1}{2}}}L(2s+1,\mathrm{sym}^2f)\mathcal{G}(2s+1;v), \tag{5.10}$$

[6]Since we require estimates that are uniform in k, the quadratic large sieve would provide a worse bound here.

where $\mathcal{G}(2s+1;v)$ is an Euler product that extends to a holomorphic function in the domain $\mathrm{Re}(s) > -1/4$ and in this region is bounded by $\ll_\varepsilon v^\varepsilon$. Using (5.10) in the integral in (5.9), shifting contours to $\mathrm{Re}(s) = -\frac{1}{4}+\varepsilon$, and using the GLH bound $L(2s+1, \mathrm{sym}^2 f) \ll_\varepsilon ((1+|s|)k)^\varepsilon$, we conclude that the $j=0$ term in (5.7) equals

$$\frac{(\frac{\eta_1}{2})^a D\lambda_f(u_1)}{8\sqrt{u_1}}\bigg(\int_{\mathbb{R}} \phi(\xi)\,\mathrm{d}\xi\bigg)L_{f,\eta}(\tfrac{1}{2})L(1,\mathrm{sym}^2 f)\mathcal{G}(1;u) + O_{\varepsilon,\phi}\bigg((ukD)^\varepsilon\bigg(\frac{D}{\sqrt{Y}}+\frac{D^{3/4}}{k^{1/4}}\bigg)\bigg). \tag{5.11}$$

It remains to estimate the contribution from the terms with $j \neq 0$ in (5.7). We split our estimate into two cases depending on whether $|j| \geq J$ where $J := \alpha^2 16vk(Dk)^\varepsilon$. By (5.6), $\widehat{F}(\frac{j}{16nv}, \frac{D}{\alpha^2}, \frac{n}{\alpha^{2+2/\delta}})$ decays rapidly when $|j| \geq J$ and adapting the argument given in [RlS15, §10.3] we get that the contribution of the terms with with $|j| \geq J$ to the right-hand side of (5.7) is $\ll (kD)^{-100}$.

Finally, we consider the terms in (5.7) the terms with $0 < |j| < J$. First we express the additive character $e(j\eta_1\overline{nv}/16)$ in terms of Dirichlet characters modulo 16, using orthogonality of characters as in [RlS15, p. 1065], to see that these terms are bounded by

$$\ll \frac{1}{\sqrt{v}}\sum_{0<|j|<J}\sum_{\alpha\leq\sqrt{Y}}\sum_{\psi \pmod{16}}\bigg|\sum_{\substack{n\geq 1\\(n,2\alpha)=1}}\frac{\lambda_f(n)}{n}\frac{\tau_j(nv)}{\sqrt{nv}}\psi(n)\widehat{F}\bigg(\frac{j}{16nv};\frac{D}{\alpha^2},\frac{n}{\alpha^{2+2/\delta}}\bigg)\bigg|. \tag{5.12}$$

Write $\Phi(s) := \int_0^\infty \phi(x)x^s\,\mathrm{d}x$ and let

$$\widetilde{F}(s,j,\alpha^2) = \int_0^\infty \widehat{F}\bigg(\frac{j}{16tu};\frac{D}{\alpha^2},\frac{t}{\alpha^{2+2/\delta}}\bigg)t^{s-1}\,\mathrm{d}t = \frac{D^{1+s}k^s}{\alpha^2}\Phi(s)\int_0^\infty V\bigg(\frac{1}{y}\bigg)e\bigg(\frac{-jy}{16v\alpha^2}\bigg)\frac{\mathrm{d}y}{y^{s+1}}. \tag{5.13}$$

Applying Mellin inversion, the sum over n in (5.12) is

$$\frac{1}{2\pi i}\int_{(2)}\widetilde{F}(s,j,\alpha^2)\bigg(\sum_{\substack{n\geq 1\\(n,2\alpha)=1}}\frac{\lambda_f(n)}{n^{1+s}}\frac{\tau_j(nv)}{\sqrt{n}}\psi(n)\bigg)\mathrm{d}s. \tag{5.14}$$

In (5.4) shifting the contour to the left, we get $V(\xi) = L_{f,\eta}(\frac{1}{2}) + O_\varepsilon(\xi^{1/2-\varepsilon})$ as $\xi \longrightarrow 0$. Using this we see that the function $\widetilde{F}$ admits an analytic continuation to $\mathrm{Re}(s) \geq -\frac{1}{2}$ and furthermore for $-\frac{1}{2}+\varepsilon \leq \mathrm{Re}(s) \leq 2$, any nonnegative integer A, and $j \in \mathbb{Z}\setminus\{0\}$ that

$$\widetilde{F}(s,j,\alpha^2) \ll_{\phi,A} \frac{D^{1+\mathrm{Re}(s)}k^{\mathrm{Re}(s)}}{\alpha^2}\bigg(\bigg(\frac{|j|}{v\alpha^2 k}\bigg)^{\mathrm{Re}(s)}+1\bigg)\bigg(\frac{1}{1+|s|}\bigg)^A. \tag{5.15}$$

The sum over n in (5.14) equals $L(1+s, f\otimes\chi_j\psi)$ times a certain Euler product which is $\ll_\varepsilon v(kD)^\varepsilon$ for $\mathrm{Re}(s) \geq -\frac{1}{2}+\varepsilon$. Additionally, GLH implies $|L(s, f\otimes\chi_j\psi)| \ll_N ((1+|s|)jkN)^\varepsilon$ for $\mathrm{Re}(s) \geq \frac{1}{2}$. We now use these bounds along with (5.15) and shift the contour of integration in (5.14) to $\mathrm{Re}(s) = -\frac{1}{2}+\varepsilon$ to bound (5.14) and conclude that the sum over n in (5.12) is, for each $0 < |j| < J$,

$$\ll_{\phi,\varepsilon} v(kD)^\varepsilon\frac{D^{1/2}k^{-1/2}}{\alpha^2}\bigg(\frac{v\alpha^2 k}{|j|}\bigg)^{1/2}.$$

Hence, we conclude that the contribution from the terms with $0 < |j| < J$ to (5.7) is

$$\ll_{\phi,\varepsilon} (kD)^\varepsilon D^{1/2}k^{-1/2}\sum_{\alpha\leq\sqrt{Y}}\frac{1}{\alpha^2}\sum_{0<|j|\leq J}\bigg(\frac{v\alpha^2 k}{|j|}\bigg)^{1/2} \ll_{\phi,\varepsilon} v(DkY)^{1/2+\varepsilon}. \tag{5.16}$$

Combining (5.5), (5.11), and (5.16), the left-hand side of (5.1) equals

$$\frac{(\frac{\eta}{2})^a D\lambda_f(u_1)}{8\sqrt{u_1}}\bigg(\int_0^\infty \phi(\xi)\,\mathrm{d}\xi\bigg)L_{f,\eta}(\tfrac{1}{2})L(1,\mathrm{sym}^2 f)\mathcal{G}(1;u)+O_{\phi,\varepsilon}\bigg((ukD)^\varepsilon\bigg(\frac{D}{\sqrt{Y}}+\frac{D^{3/4}}{k^{1/4}}+u(YkD)^{1/2}\bigg)\bigg).$$

To balance error terms we take $Y = D^{1/2}/(uk^{1/2})$, which completes the proof. □

5.2. **The case $g = 1$ of Proposition 3.2.** For the rest of this paper let $F \in S_k(\Gamma)$ traverse a sequence of Saito–Kurokawa lifts that are Hecke eigenforms. We freely use the notations from Sections 3.2 and 3.3. The goal of this subsection is to prove that

$$\frac{1}{\|F\|_2^2}\sum_{T\in\Lambda_2^+/\mathrm{SL}_2(\mathbb{Z})}\frac{|R(T)|^2}{\varepsilon(T)}|\operatorname{disc} T|^{k-3/2}G(T;1,\kappa)\longrightarrow\frac{\mathrm{vol}(\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H})}{2\cdot\mathrm{vol}(\Gamma\backslash\mathbb{H}_2)}\widetilde{\kappa}(3) \tag{5.17}$$

as $k\longrightarrow\infty$.

Observe that by Mellin inversion and writing $Y = \lambda g_z {}^t g_z$, we have

$$\begin{aligned}G(T;1,\kappa) &= \frac{1}{2}\int_{M_2^{\mathrm{Sym}}(\mathbb{R})^+}\kappa(\sqrt{\det Y})(\det Y)^{k-3}e^{-4\pi\mathrm{Tr}(TY)}\,\mathrm{d}Y\\ &= \frac{1}{2}\cdot\frac{1}{2\pi i}\int_{(\sigma)}\widetilde{\kappa}(s)\int_{M_2^{\mathrm{Sym}}(\mathbb{R})^+}(\det Y)^{k-3+s/2}e^{-4\pi\mathrm{Tr}(TY)}\mathrm{d}Y\,\mathrm{d}s\end{aligned}$$

for any $\sigma > 2$.

To evaluate the inner integral we recognize it as a value of Siegel's generalized Gamma function [Sie35, Hilfssatz 3] to see that the integral equals

$$\sqrt{\pi}\,(\det T)^{-k-s/2+3/2}\left(\frac{1}{4\pi}\right)^{2k+s-3}\Gamma\left(k+\frac{s}{2}-\frac{3}{2}\right)\Gamma\left(k+\frac{s}{2}-2\right).$$

We also change the discriminants to determinants by recalling the relation $\operatorname{disc}(T) = -4\det T$. Combining these observations lead to

$$\begin{aligned}&|\operatorname{disc}(T)|^{k-3/2}G(T;1,\kappa)\\ &\quad=\frac{4^{-k+3/2}\pi^{-2k+7/2}}{2}\cdot\frac{1}{2\pi i}\int_{(\sigma)}\widetilde{\kappa}(s)(\pi^2\det T)^{-s/2}\Gamma\left(k+\frac{s}{2}-2\right)\Gamma\left(k+\frac{s}{2}-\frac{3}{2}\right)ds.\end{aligned}$$

For $s\in\mathbb{C}$ with $\mathrm{Re}(s) > 3/2$, we define the Rankin–Selberg convolution of the Koecher–Maass series

$$D(s) := \sum_{T\in\Lambda_2^+/\mathrm{SL}_2(\mathbb{Z})}\frac{|R(T)|^2}{\varepsilon(T)(4\det T)^s}.$$

It is known [Kal83] that $D(s)$ has a pole at $s = 3/2$ and can be meromorphically continued to the whole complex plane. So (5.17) is equivalent to showing that

$$\begin{aligned}&\frac{4^{-k+3/2}\pi^{-2k+7/2}}{2\|F\|_2^2}\cdot\frac{1}{2\pi i}\int_{(\sigma)}\widetilde{\kappa}(s)(4\pi^2)^{-s/2}D(s/2)\Gamma\left(k+\frac{s}{2}-\frac{3}{2}\right)\Gamma\left(k+\frac{s}{2}-2\right)\mathrm{d}s\\ &\quad\longrightarrow\frac{\mathrm{vol}(\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H})}{2\cdot\mathrm{vol}(\Gamma\backslash\mathbb{H}_2)}\widetilde{\kappa}(3)\end{aligned} \tag{5.18}$$

as $k\longrightarrow\infty$.

We modify the left-hand side of (5.18) by Stirling's formula. Using the approximation for $\Gamma(s/2+k-3/2)/\Gamma(k-3/2)$ and $\Gamma(k+s/2-2)/\Gamma(k-2)$ we see that the left-hand side of (5.18) as $k \longrightarrow \infty$ is

$$\sim \frac{4^{-k+3/2}\pi^{-2k+7/2}}{2\|F\|_2^2}\Gamma\left(k-\frac{3}{2}\right)\Gamma(k-2)\cdot\left(\frac{1}{2\pi i}\int\limits_{(\sigma)}\widetilde{\kappa}(s)(4\pi^2)^{-s/2}D(s/2)\left(k-\frac{3}{2}\right)^{s/2}(k-2)^{s/2}\,\mathrm{d}s\right)$$

$$= \frac{4^{-k+3/2}\pi^{-2k+7/2}}{2\|F\|_2^2}\Gamma\left(k-\frac{3}{2}\right)\Gamma(k-2)$$

$$\times \sum_{T\in\Lambda_2^+/\mathrm{SL}_2(\mathbb{Z})}\frac{|R(T)|^2}{\varepsilon(T)}\cdot\frac{1}{2\pi i}\int\limits_{(\sigma)}\widetilde{\kappa}(s)(4\pi^2|\mathrm{disc}\ (T)|)^{-s/2}\left(k-\frac{3}{2}\right)^{s/2}(k-2)^{s/2}\,\mathrm{d}s$$

(5.19)

$$= \frac{4^{-k+3/2}\pi^{-2k+7/2}}{2\|F\|_2^2}\Gamma\left(k-\frac{3}{2}\right)\Gamma(k-2)\sum_{T\in\Lambda_2^+/\mathrm{SL}_2(\mathbb{Z})}\frac{|R(T)|^2}{\varepsilon(T)}\kappa\left(\sqrt{\frac{\left(k-\frac{3}{2}\right)(k-2)}{4\pi^2|\mathrm{disc}\ (T)|}}\right),$$

where we have used Mellin inversion in the last step. The inner sum will be estimated by the following result.

Proposition 5.2. *Assume GLH. Let $W \in C_c^\infty(\mathbb{R}_{>0})$. Then for any $\varepsilon > 0$ we have that*

(5.20)

$$\sum_{T\in\Lambda_2^+/\mathrm{SL}_2(\mathbb{Z})}\frac{|R(T)|^2}{\varepsilon(T)}W\left(\frac{|\,\mathrm{disc}(T)|}{D}\right)$$

$$= \left(\mathrm{Res}_{s=3/2}\,D(s)\right)\left(\int\limits_0^\infty\sqrt{\xi}W(\xi)\,\mathrm{d}\xi\right)\left(D^{3/2}+O_{\varepsilon,W}\left((kD)^\varepsilon D^{3/2-1/8}k^{1/8}\right)\right).$$

We prove Proposition 5.2 further below. Also, note that from the proofs of Propositions 5.1 and 5.2 it is clear that the error term in (5.20) only depends on at most ε and $\|W^{(j)}\|_\infty$, $j = 0, 1, \ldots$. Assuming the truth of Proposition 5.2 for now, we apply the result with the choices $D = k^2$ and

$$W(\xi) = \kappa\left(\sqrt{\frac{\left(k-\frac{3}{2}\right)(k-2)}{4\pi^2\xi k^2}}\right),$$

so that $\|W^{(j)}\|_\infty \ll \|\kappa^{(j)}\|_\infty \ll_{j,\kappa} 1$. We also have that

$$\int\limits_0^\infty\sqrt{\xi}W(\xi)\,\mathrm{d}\xi = \frac{(k-2)^{3/2}\left(k-\frac{3}{2}\right)^{3/2}}{4\pi^3k^3}\widetilde{\kappa}(3),$$

and conclude that the sum (5.19) equals

$$\frac{4^{-k+3/2}\pi^{-2k+7/2}}{2\|F\|_2^2}\Gamma\left(k-\tfrac{3}{2}\right)\Gamma(k-2)\left(\mathrm{Res}_{s=3/2}\,D(s)\right)\frac{(k-2)^{3/2}\left(k-\frac{3}{2}\right)^{3/2}}{4\pi^3k^3}\widetilde{\kappa}(3)\cdot\left(k^3+O\left(k^{23/8}\right)\right).$$

Using Stirling's approximation this simplifies further to

$$\frac{4^{-k}\pi^{-2k+1/2}}{\|F\|_2^2}\Gamma\left(k-\frac{1}{2}\right)\Gamma(k)\left(\mathrm{Res}_{s=3/2}\,D(s)\right)\widetilde{\kappa}(3)\left(1+O\left(\frac{1}{k^{1/8}}\right)\right). \tag{5.21}$$

Noting that

$$D(s) = 4^{-k-s+3/2} \sum_{T \in \Lambda_2^+/\mathrm{SL}_2(\mathbb{Z})} \frac{|a(T)|^2}{\varepsilon(T)(\det T)^{s+k-3/2}}$$

we compute the residue of $D(s)$ at the simple pole $s = 3/2$ from previous works of Kalinin [Kal83] and Katsurada–Kim [KK22]. From these papers it follows that

$$\mathrm{Res}_{s=3/2} D(s) = 4^{-k} \cdot \frac{\pi^{-3/2}\Gamma(3/2)\zeta(3)\|F\|_2^2}{2^{1-4k}\pi^{-2k-3}\Gamma(k)\Gamma(k-1/2)\Gamma(3/2)\Gamma(2)\zeta(3)\zeta(4)} \tag{5.22}$$

$$= \frac{4^{k-1/2}\pi^{2k+3/2}\|F\|_2^2}{\Gamma(k)\Gamma\left(k-\frac{1}{2}\right)\zeta(4)}. \tag{5.23}$$

Combining the above computations gives that (5.21) is

$$\sim \frac{4^{-k}\pi^{-2k+1/2}}{\|F\|_2^2}\Gamma\left(k-\frac{1}{2}\right)\Gamma(k)\frac{4^{k-1/2}\pi^{2k+3/2}\|F\|_2^2}{\Gamma(k)\Gamma\left(k-\frac{1}{2}\right)\zeta(4)}\widetilde{\kappa}(3) = \frac{\pi^2}{2\zeta(4)}\widetilde{\kappa}(3) = \frac{45}{\pi^2}\widetilde{\kappa}(3)$$

as $k \longrightarrow \infty$.

Now to get (5.18) it is enough to note that

$$\frac{\mathrm{vol}(\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H})}{2 \cdot \mathrm{vol}(\Gamma\backslash\mathbb{H}_2)} = \frac{\pi/3}{2 \cdot \pi^3/270} = \frac{45}{\pi^2},$$

where we have used [Sie43] to compute the volume of the Siegel modular variety.

We finish the analysis of the constant term contribution by proving Proposition 5.2.

Proof of Proposition 5.2. Assume $D \geq k$, since for $D < k$ the result is an easy consequence of Lemma 4.13 under GLH. Also, since $W \in C_c^\infty(\mathbb{R}_{>0})$ we will restrict to T with $\varepsilon(T) = 2$ as this is true for all T with $|\mathrm{disc}(T)| > 4$. Given $T \in \Lambda_2^+/\mathrm{SL}_2(\mathbb{Z})$ we write $\mathrm{disc}(T) = h^2 d$ with $d \in \mathcal{D}$. We have that

$$\sum_{T \in \Lambda_2^+/\mathrm{SL}_2(\mathbb{Z})} |R(T)|^2 W\left(\frac{|\mathrm{disc}(T)|}{D}\right) = \sum_{h \in \mathbb{N}} \sum_{d \in \mathcal{D}} \sum_{\substack{T \in \Lambda_2^+/\mathrm{SL}_2(\mathbb{Z}) \\ \mathrm{disc}(T) = h^2 d}} |R(T)|^2 W\left(\frac{h^2|d|}{D}\right). \tag{5.24}$$

Let $1 \leq H \leq (D/k)^{1/2}$. Using Lemma 4.13 and applying GLH to $L(s, f \otimes \chi_d)$ we get that the contribution from the terms with $h \geq H$ is $\ll_{\varepsilon,W} (Dk)^\varepsilon \|F\|_2^2 D^{3/2}/(c_k H)$, where we also used the bound $L(1, \mathrm{sym}^2 f) \gg k^{-\varepsilon}$ due to Hoffstein and Lockhart [HL94].

For the terms with $1 \leq h \leq H$, we write $T = gT'$ where $g = \mathrm{cont}(T) = (m, n, r)$ so that $\mathrm{cont}(T') = (m/g, n/g, r/g) = 1$. Applying the definition of R given in (3.3) and (4.23) with $a = h/g$ we have that

$$\begin{aligned} &\sum_{1 \leq h \leq H} \sum_{d \in \mathcal{D}} \sum_{\substack{T \in \Lambda_2^+/\mathrm{SL}_2(\mathbb{Z}) \\ \mathrm{disc}(T) = h^2 d}} |R(T)|^2 W\left(\frac{h^2|d|}{D}\right) \\ &= \sum_{1 \leq h \leq H} \sum_{d \in \mathcal{D}} \sum_{g|h} \left| \sum_{j|g} c\left(\frac{h^2|d|}{j^2}\right)\sqrt{j} \right|^2 \frac{\sqrt{|d|}h}{g\pi} L(1, \chi_d) \sum_{t|(h/g)} \frac{\mu(t)\chi_d(t)}{t} W\left(\frac{h^2|d|}{D}\right). \end{aligned} \tag{5.25}$$

By (3.5) we may write

$$|c(|d|)|^2 = C_{\tilde{f}} L(\tfrac{1}{2}, f \otimes \chi_d),$$

where

$$C_{\tilde{f}} = \frac{\|\widetilde{f}\|_2^2}{L(1, \mathrm{sym}^2 f)} \frac{2^{2k-2}\pi^{k+\frac{1}{2}}}{\Gamma(k-\frac{1}{2})} \asymp \frac{\|F\|_2^2}{c_k L(1, \mathrm{sym}^2 f)}$$

and c_k is given in (3.6). Using this along with (4.24) we have that

$$\begin{aligned}&\Big|\sum_{j|g} c\Big(\frac{h^2|d|}{j^2}\Big)\sqrt{j}\Big|^2\\ &= C_{\tilde f}L(\tfrac12, f\otimes\chi_d)\sum_{[j_1,j_2]|g}\sqrt{j_1j_2}\sum_{u_1v_1=h/j_1}\frac{\mu(u_1)\chi_d(u_1)}{\sqrt{u_1}}\lambda_f(v_1)\sum_{u_2v_2=h/j_2}\frac{\mu(u_2)\chi_d(u_2)}{\sqrt{u_2}}\lambda_f(v_2).\end{aligned}\tag{5.26}$$

Also, it is not hard to see that GLH for $L(s,\chi_d)$ implies that

$$L(1,\chi_d)=\sum_{a\le H^2}\frac{\chi_d(a)}{a}+O_\varepsilon(|d|^\varepsilon H^{-1+\varepsilon}).\tag{5.27}$$

Recall $\mathcal{R}=\{1,5,8,9,12,13\}$. Using (5.26) and (5.27) we get that the right-hand side of (5.25) is

$$\begin{aligned}&\frac{C_{\tilde f}\sqrt{D}}{\pi}\sum_{\eta\in\mathcal{R}}\sum_{1\le h\le H}\sum_{g|h}\frac1g\sum_{[j_1,j_2]|g}\sqrt{j_1j_2}\sum_{u_1v_1=h/j_1}\frac{\mu(u_1)}{\sqrt{u_1}}\lambda_f(v_1)\sum_{u_2v_2=h/j_2}\frac{\mu(u_2)}{\sqrt{u_2}}\lambda_f(v_2)\sum_{a\le H^2}\frac1a\\ &\times\sum_{t|(h/g)}\frac{\mu(t)}{t}\sum_{d\in\mathcal{D}_\eta}L(\tfrac12,f\otimes\chi_d)\chi_d(tu_1u_2a)\Big(\frac{h\sqrt{|d|}}{\sqrt{D}}W\Big(\frac{h^2|d|}{D}\Big)\Big)+O_{\varepsilon,W}\Big(C_{\tilde f}(Dk)^\varepsilon\frac{D^{3/2}}{H^{1-\varepsilon}}\Big).\end{aligned}\tag{5.28}$$

Applying Proposition 5.1 with $\phi(\xi)=\sqrt{\xi}W(\xi)$ and writing $b=tu_1u_2a=2^cb_1b_2^2$, where b_1 is odd and squarefree, we have that the innermost sum in (5.28) equals

$$\Big(\frac{-\eta}{2}\Big)^c\frac{D\lambda_f(b_1)}{8h^2\sqrt{b_1}}\Big(\int_0^\infty\sqrt{\xi}W(\xi)\,\mathrm{d}\xi\Big)L_{f,\eta}(\tfrac12)L(1,\mathrm{sym}^2 f)\mathcal{G}(1;b)+O_{\varepsilon,W}\Big((bkD)^\varepsilon\sqrt{b}\frac{D^{3/4}}{h^{3/2}}k^{1/4}\Big).$$

We now use this formula in (5.28) and complete the sums over h,a in the main term to sums over all positive integers at the cost of an error term of size $\ll_{\varepsilon,W} C_{\tilde f}(Dk)^\varepsilon D^{3/2}/H$ to get that the right-hand side of (5.25) is

$$\begin{aligned}&\frac{C_{\tilde f}D^{3/2}L(1,\mathrm{sym}^2 f)}{8\pi}\Big(\int_0^\infty\sqrt{\xi}W(\xi)\,\mathrm{d}\xi\Big)\sum_{\eta\in\mathcal{R}}L_{f,\eta}(\tfrac12)\sum_{h\ge1}\frac{1}{h^2}\sum_{g|h}\sum_{[j_1,j_2]|g}\sqrt{j_1j_2}\\ &\times\sum_{u_1v_1=h/j_1}\frac{\mu(u_1)}{\sqrt{u_1}}\lambda_f(v_1)\sum_{u_2v_2=h/j_2}\frac{\mu(u_2)}{\sqrt{u_2}}\lambda_f(v_2)\sum_{t|(h/g)}\frac{\mu(t)}{t}\sum_{a\ge1}\frac{\lambda_f(b_1)\mathcal{G}(1;b)}{a\sqrt{b_1}}\Big(\frac{-\eta}{2}\Big)^c\\ &+O_{\varepsilon,W}\Big(C_{\tilde f}(Dk)^\varepsilon\Big(\frac{D^{3/2}}{H}+HD^{5/4}k^{1/4}\Big)\Big).\end{aligned}\tag{5.29}$$

We now choose $H=(D/k)^{1/8}$ so that the error term is $\ll_{\varepsilon,W} C_{\tilde f}(Dk)^\varepsilon D^{3/2-1/8}k^{1/8}$.

To complete the proof, note by [Kal83] that $D(s)$ has a simple pole at $s=3/2$ and admits a meromorphic continuation to the complex plane, furthermore as $D\longrightarrow\infty$

$$\sum_{T\in\Lambda_2^+/\mathrm{SL}_2(\mathbb{Z})}\frac{|R(T)|^2}{\varepsilon(T)}W\Big(\frac{\mathrm{disc(T)}}{D}\Big)\sim\Big(\mathrm{Res}_{s=3/2}D(s)\Big)\Big(\int_0^\infty\sqrt{\xi}W(\xi)\,\mathrm{d}\xi\Big)D^{3/2},\tag{5.30}$$

(assuming W is not identically 0). Comparing the main term in (5.29) with (5.30), i.e. fix f and take $D\longrightarrow\infty$ in (5.29), the leading order constants must match for any given f so we conclude

that

$$\sum_{T\in\Lambda_2^+/\mathrm{SL}_2(\mathbb{Z})}\frac{|R(T)|^2}{\varepsilon(T)}W\Big(\frac{|\operatorname{disc}(T)|}{D}\Big)=\Big(\operatorname{Res}_{s=3/2}D(s)\Big)\Big(\int_0^\infty\sqrt{\xi}W(\xi)\,\mathrm{d}\xi\Big)D^{3/2}+O_{\varepsilon,W}\Big(C_{\tilde f}(Dk)^\varepsilon D^{3/2-1/8}k^{1/8}\Big). \tag{5.31}$$

Also, since $C_{\tilde f}\asymp\|F\|_2^2/(c_kL(1,\operatorname{sym}^2 f))$ we have by (5.22) and (3.6) that

$$\Big(\operatorname{Res}_{s=3/2}D(s)\Big)\asymp L(1,\operatorname{sym}^2 f)C_{\tilde f}. \tag{5.32}$$

Recalling that $L(1,\operatorname{sym}^2 f)\gg_\varepsilon k^{-\varepsilon}$ and using (5.32) in (5.31) completes the proof. □

As stated above, this finishes the proof of (5.17).

5.3. **The weight function as a period integral.** We now embark on the task of proving Proposition 3.2 for functions g that are orthogonal to the constant function. For this, we will take an average of the weight function over a class group and then reinterpret part of the resulting integral as a period over a non-split torus.

Let $D<0$ be a discriminant. For a positive integer L, we let $H(D;L)$ denote the set of $\mathrm{SL}_2(\mathbb{Z})$-equivalence classes of matrices in Λ_2 such that $\operatorname{cont}(T)=L$ and $\operatorname{disc}(T)=DL^2$. It is easy to see that map $S\mapsto L^{-1}S$ gives a bijection $H(D;L)\simeq H(D;1)$ and it is a classical fact going back to Gauss that the latter set can be naturally identified with the class group of the unique order of discriminant D in $\mathbb{Q}(\sqrt{d})$. We denote

$$h(D)=|H(D;L)|=|H(D;1)|.$$

In particular, if $D=d$ is a fundamental discriminant, then $h(d)$ is the class number of $\mathbb{Q}(\sqrt{d})$. If D is not fundamental, we may write $D=dM^2$ with $d\in\mathcal{D}$ a fundamental discriminant and in this case we have the formula [Coh80, p. 217]

$$h(D)=\frac{M}{u(d)}\,h(d)\prod_{p\mid M}\Big(1-\Big(\frac{d}{p}\Big)p^{-1}\Big),$$

where $u(-3)=3$, $u(-4)=2$ and $u(d)=1$ for other d.

Recall that the quantity $G(S;g,\kappa)$ defined in (3.8) depends only on the $\mathrm{SL}_2(\mathbb{Z})$-equivalence class of S. Therefore, for D,L as above, a slowly growing function $g:\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}\longrightarrow\mathbb{C}$, and $\kappa\in C_c^\infty(\mathbb{R}^+)$ the following is well-defined.

$$\begin{aligned}G(D;L;g,\kappa)&:=\frac{|DL^2|^{k-\frac32}}{h(D)}\sum_{S\in H(D;L)}G(S;g,\kappa)\\&=\frac{|DL^2|^{k-\frac32}}{h(D)}\sum_{S\in H(D;L)}\int_0^\infty\int_{z=u+iy\in\mathbb{H}}g(z)\lambda^{2k}\kappa(\lambda)e^{-4\pi\lambda\mathrm{Tr}(Sg_z{}^tg_z)}\frac{\mathrm{d}u\,\mathrm{d}y\,\mathrm{d}\lambda}{y^2\lambda^4}.\end{aligned} \tag{5.33}$$

Moreover, since $R(T)$ depends only on $\operatorname{cont}(T)$ and $\operatorname{disc}(T)$, and $\varepsilon(T)$ depends only on D, we define

$$R(D;L):=R(T),\quad \varepsilon(D)=\varepsilon(T)$$

for any T satisfying $\operatorname{cont}(T)=L,\ \operatorname{disc}(T)=DL^2$. We note for future reference that

$$\sum_{T\in\Lambda_2/\mathrm{SL}_2(\mathbb{Z})}\frac{|R(T)|^2}{\varepsilon(T)}|\operatorname{disc}(T)|^{k-\frac32}G(T;g,\kappa)=\sum_{L,D}\frac{|R(D;L)|^2}{\varepsilon(D)}|DL^2|^{k-\frac32}\sum_{T\in H(D;L)}G(T;g,\kappa) \tag{5.34}$$

$$= \sum_{L,D} \frac{h(D)|R(D;L)|^2}{\varepsilon(D)} G(D;L;g,\kappa), \tag{5.35}$$

where L ranges over the positive integers and D ranges over the set of negative discriminants, i.e., $D<0,\ D\equiv 0,1\,(4)$.

For each discriminant $D=dM^2$ where $d<0$ is a fundamental discriminant, we will now rewrite $G(D;L;g,\kappa)$ as a certain period integral. Let

$$S_d = \begin{pmatrix} a_d & b_d \\ b_d & 1 \end{pmatrix} := \begin{cases} \left[\begin{smallmatrix} \frac{-d}{4} & 0 \\ 0 & 1 \end{smallmatrix}\right] & \text{if } d\equiv 0\,(4), \\ \left[\begin{smallmatrix} \frac{1-d}{4} & \frac{1}{2} \\ \frac{1}{2} & 1 \end{smallmatrix}\right] & \text{if } d\equiv 1\,(4). \end{cases} \tag{5.36}$$

Given S_d as above, one obtains a non-split torus T_d embedded in GL_2. Precisely, for each ring R, we set

$$T_d(R) := \{g\in \mathrm{GL}_2(R) :\ {}^t g S_d g = \det(g) S_d\}. \tag{5.37}$$

We have $T_d(\mathbb{Q})\simeq K^\times$ where $K=\mathbb{Q}(\sqrt{d})$ via

$$\begin{pmatrix} x+yb_d/2 & y \\ -ya_d & x-yb_d/2 \end{pmatrix} \longmapsto x+y\frac{\sqrt{d}}{2}. \tag{5.38}$$

We define

$$\mathrm{Cl}_D = T_d(\mathbb{A})/T_d(\mathbb{Q})T_d(\mathbb{R})U_T(M),$$

where $U_T(M) := \prod_{p<\infty} U_{T_d,p}(m_p)$ with $M=\prod_{p<\infty} p^{m_p}$ and the subgroup $U_{T_d,p}(m)\subset T_d(\mathbb{Z}_p)$ is defined via

$$U_{T_d,p}(m) := \left\{ g\in T_d(\mathbb{Z}_p) : g\equiv \begin{pmatrix} \lambda & \\ & \lambda \end{pmatrix} (p^m) \text{ for some } \lambda\in \mathbb{Z}_p^\times \right\}.$$

For each $c\in \mathrm{Cl}_D$, pick $t_c\in \prod_{p<\infty} T_d(\mathbb{Q}_p)$ such that

$$T_d(\mathbb{A}) = \bigsqcup_{c\in \mathrm{Cl}_D} t_c T_d(\mathbb{Q}) T_d(\mathbb{R}) U_T(M).$$

By strong approximation, write $t_c=\gamma_c m_c \kappa_c$ with $\gamma_c\in \mathrm{GL}_2(\mathbb{Q})$, $m_c\in \mathrm{GL}_2(\mathbb{R})^+$, and $\kappa_c\in U_T(M)$; note that $(\gamma_c)_\infty = m_c^{-1}$. The matrices

$$S_c = (\det \gamma_c)^{-1}\ {}^t\gamma_c S \gamma_c$$

satisfy $\mathrm{cont}(S_c)=1$, $\mathrm{disc}(S_c)=d$. Also, the lower right entry of S_c is 1 modulo M. For any positive integer L, we define

$$\phi_{L,M}(c) := \begin{pmatrix} L & \\ & L \end{pmatrix} \begin{pmatrix} M & \\ & 1 \end{pmatrix} S_c \begin{pmatrix} M & \\ & 1 \end{pmatrix}. \tag{5.39}$$

It follows that $\mathrm{cont}(\phi_{L,M}(c))=L$, $\mathrm{disc}(\phi_{L,M}(c))=DL^2$. By Prop 5.3 of [PSS17] the map $c\mapsto [\phi_{L,M}(c)]$ gives a bijection from Cl_D to $H(D;L)$.

For $z=u+iy$ put $dz=\frac{dudy}{y^2}$ and write $a_M=\begin{pmatrix} M^{1/2} & \\ & M^{-1/2} \end{pmatrix}$. Using the above discussion, we can write (5.33) as

$$\begin{aligned} G(D;L;g,\kappa) &= \frac{|DL^2|^{k-\frac{3}{2}}}{h(D)} \sum_{S\in H(D;L)} \int_{\mathbb{R}^+}\int_{\mathbb{H}} g(z)\lambda^{2k}\kappa(\lambda) e^{-4\pi\lambda \mathrm{Tr}(S g_z {}^t g_z)} \frac{\mathrm{d}z\,\mathrm{d}\lambda}{\lambda^4} \\ &= \frac{|DL^2|^{k-\frac{3}{2}}}{h(D)} \sum_{c\in \mathrm{Cl}_D} \int_{\mathbb{R}^+}\int_{\mathbb{H}} g(z)\lambda^{2k}\kappa(\lambda) e^{-4\pi\lambda LM \mathrm{Tr}(\det(\gamma_c)^{-1}\ a_M({}^t\gamma_c) S_d \gamma_c a_M g_z {}^t g_z)} \frac{\mathrm{d}z\,\mathrm{d}\lambda}{\lambda^4} \end{aligned}$$

$$= \frac{|DL^2|^{k-\frac{3}{2}}}{h(D)} \sum_{c\in \mathrm{Cl}_D} \int\limits_{\mathbb{R}^+} \int\limits_{\mathbb{H}} g(z)\lambda^{2k}\kappa(\lambda) e^{-4\pi\lambda LM \mathrm{Tr}(\det(\gamma_c)^{-1}\ S_d \gamma_c a_M g_z {}^t g_z a_M ({}^t\gamma_c))} \frac{\mathrm{d}z\,\mathrm{d}\lambda}{\lambda^4}$$

$$= \frac{|DL^2|^{k-\frac{3}{2}}}{h(D)} \sum_{c\in \mathrm{Cl}_D} \int\limits_{\mathbb{R}^+} \int\limits_{\mathbb{H}} g(z)\lambda^{2k}\kappa(\lambda) e^{-4\pi\lambda LM \mathrm{Tr}(\det(\gamma_c)^{-1}\ S_d g_{(\gamma_c Mz)} {}^t g_{(\gamma_c Mz)} a_M ({}^t\gamma_c))} \frac{\mathrm{d}z\,\mathrm{d}\lambda}{\lambda^4}$$

$$= \frac{|DL^2|^{k-\frac{3}{2}}}{h(D)} \int\limits_{\mathbb{R}^+} \int\limits_{\mathbb{H}} \sum_{c\in \mathrm{Cl}_D} g(M^{-1}\gamma_c^{-1} z)\lambda^{2k}\kappa(\lambda) e^{-4\pi\lambda LM \mathrm{Tr}(S_d g_z {}^t g_z)} \frac{\mathrm{d}z\,\mathrm{d}\lambda}{\lambda^4}.$$

Let $T_d^1(\mathbb{R}) := \{g \in T_d(\mathbb{R}) : \det(g) = 1\}$. Then we have an isomorphism $\{\pm 1\}\backslash T_d^1(\mathbb{R}) \simeq \mathbb{R}^\times \backslash T(\mathbb{R})$. Writing elements of $\mathbb{H}$ as tz_0 where $t \in T_d^1(\mathbb{R})$ and $z_0 \in T_d^1(\mathbb{R})\backslash\mathbb{H}$, we have $G(D; L; g, \kappa)$ equals

$$|DL^2|^{k-\frac{3}{2}} \int\limits_{\mathbb{R}^+} \int\limits_{T_d^1(\mathbb{R})\backslash\mathbb{H}} \sum_{c\in \mathrm{Cl}_D} \frac{1}{h(D)} \int\limits_{T_d^1(\mathbb{R})} g(M^{-1}\gamma_c^{-1} tz_0)\lambda^{2k}\kappa(\lambda) e^{-4\pi\lambda LM \mathrm{Tr}(S_d g_{z_0} {}^t g_{z_0})} \frac{\mathrm{d}t\,\mathrm{d}z_0\,\mathrm{d}\lambda}{\lambda^4},$$

where we have used crucially the fact that ${}^t t S_d t = S_d$ for all $t \in T_d^1(\mathbb{R})$.

The upshot is that

$$G(D; L; g, \kappa) = |DL^2|^{k-\frac{3}{2}} \int\limits_{\mathbb{R}^+} \lambda^{2k}\kappa(\lambda) \int\limits_{T_d^1(\mathbb{R})\backslash\mathbb{H}} W(g; D, z_0) e^{-4\pi\lambda LM \mathrm{Tr}(S_d g_{z_0} {}^t g_{z_0})} \frac{\mathrm{d}z_0\,\mathrm{d}\lambda}{\lambda^4}, \tag{5.40}$$

where

$$W(g; D, z_0) := \frac{1}{h(D)} \sum_{c\in \mathrm{Cl}_D} \int\limits_{T_d^1(\mathbb{R})} g(M^{-1}\gamma_c^{-1} tz_0)\,\mathrm{d}t.$$

We now let ϕ_g be the adelization of g, i.e., ϕ_g is the unique function on $\mathrm{GL}_2(\mathbb{A})$ satisfying

$$\phi_g(z h_{\mathbb{Q}} h_\infty k) = g(h_\infty i) \tag{5.41}$$

for all $z \in Z(\mathbb{A})$, $h_{\mathbb{Q}} \in \mathrm{GL}_2(\mathbb{Q})$, $k \in \mathrm{SO}(2)\prod_p \mathrm{GL}_2(\mathbb{Z}_p)$ and $h_\infty \in \mathrm{GL}_2(\mathbb{R})^+$. Let $k^{(M)} = (k_p^{(M)})_{p<\infty} \in \mathrm{GL}_2(\mathbb{A}_f)$ be given by

$$k_p^{(M)} := \begin{cases} \left(\begin{smallmatrix} M & \\ & 1 \end{smallmatrix}\right) & \text{if } p \mid M \\ 1 & \text{if } p \nmid M \text{ or } p = \infty \end{cases} \tag{5.42}$$

We let $\phi_g^{(M)}$ be the function on $\mathrm{GL}_2(\mathbb{A})$ by right-translation of ϕ_g by $k^{(M)}$, i.e., $\phi_g^{(M)}(h) = \phi_g(hk^{(M)})$. It is easy to see that

$$\phi_g^{(M)}(h_\infty) = g(M^{-1}h_\infty i), \qquad h_\infty \in \mathrm{GL}_2(\mathbb{R})^+.$$

The next key lemma reinterprets $W(g; D, z_0)$ as a toric period of $\phi_g^{(M)}$.

Lemma 5.3. *Let $g : \mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H} \longrightarrow \mathbb{C}$ be slowly growing and $\phi_g^{(M)}$ be defined as above. For $z_0 \in \mathbb{H}$, let $g_{z_0} \in \mathrm{GL}_2(\mathbb{R})^+$ be such that $g_{z_0} i = z_0$. Let $d < 0$ be a fundamental discriminant and $D = dM^2$. We have*

$$\int\limits_{\mathbb{A}^\times T_d(\mathbb{Q})\backslash T_d(\mathbb{A})} \phi_g^{(M)}(tg_{z_0})\,\mathrm{d}t = \frac{\mathrm{vol}(\mathbb{A}^\times T_d(\mathbb{Q})\backslash T_d(\mathbb{A}))}{\mathrm{vol}(T_d^1(\mathbb{R}))} W(g; D, z_0). \tag{5.43}$$

Proof. Noting that $\phi_g^{(M)}(t_1 t g_{z_0}) = \phi_g^{(M)}(t_1 g_{z_0})$ for all $t_1 \in T_d(\mathbb{A})$, $t \in \mathbb{A}^\times T_d(\mathbb{Q}) U_T(M)$ we obtain

$$\int\limits_{\mathbb{A}^\times T_d(\mathbb{Q})\backslash T_d(\mathbb{A})} \phi_g^{(M)}(tg_{z_0})\,\mathrm{d}t = \sum_{c\in \mathrm{Cl}_D} \int\limits_{\mathbb{A}^\times T_d(\mathbb{Q})\backslash T_d(\mathbb{Q})T_d(\mathbb{R})U_T(M)} \phi_g^{(M)}(t_c t g_{z_0})\,\mathrm{d}t$$

$$= \frac{\mathrm{vol}(\mathbb{A}^\times T_d(\mathbb{Q})\backslash T_d(\mathbb{Q})T_d(\mathbb{R})U_T(M))}{\mathrm{vol}(\mathbb{R}^\times\backslash T_d(\mathbb{R}))} \sum_{c\in \mathrm{Cl}_D} \int\limits_{\mathbb{R}^\times\backslash T_d(\mathbb{R})} \phi_g^{(M)}(t_c t g_{z_0})\,\mathrm{d}t$$

$$= \frac{\mathrm{vol}(\mathbb{A}^\times T_d(\mathbb{Q})\backslash T_d(\mathbb{Q})T_d(\mathbb{R})U_T(M))}{\mathrm{vol}(T_d^1(\mathbb{R}))} \sum_{c\in \mathrm{Cl}_D} \int\limits_{T_d^1(\mathbb{R})} \phi_g^{(M)}((\gamma_c)_\infty^{-1} t g_{z_0})\,\mathrm{d}t$$

$$= \frac{\mathrm{vol}(\mathbb{A}^\times T_d(\mathbb{Q})\backslash T_d(\mathbb{A}))}{\mathrm{vol}(T_d^1(\mathbb{R}))} \frac{1}{h(D)} \sum_{c\in \mathrm{Cl}_D} \int\limits_{T_d^1(\mathbb{R})} g(M^{-1}(\gamma_c)^{-1} t z_0)\,\mathrm{d}t$$

as required. □

We give $\mathbb{A}^\times T_d(\mathbb{Q})\backslash T_d(\mathbb{A})$ the Tamagawa measure as usual, which gives it total volume 2. We summarize the results so far.

Proposition 5.4. *Let L, M be positive integers and let $d<0$ be a fundamental discriminant; set $D=dM^2$. Let $g:\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}\longrightarrow\mathbb{C}$ be slowly growing, and $\kappa\in C_c^\infty(\mathbb{R}^+)$. Let the adelization ϕ_g of g be given by* (5.41) *and let $\phi_g^{(M)}(h)=\phi_g(hk^{(M)})$ for all $h\in\mathrm{GL}_2(\mathbb{A})$. The quantity $G(D;L;g,\kappa)$ equals*

$$|DL^2|^{k-\frac{3}{2}}\frac{\mathrm{vol}(T_d^1(\mathbb{R}))}{2}\int\limits_{\mathbb{R}^+}\lambda^{2k}\kappa(\lambda)\int\limits_{T_d^1(\mathbb{R})\backslash\mathbb{H}}\left(\int\limits_{\mathbb{A}^\times T_d(\mathbb{Q})\backslash T_d(\mathbb{A})}\phi_g^{(M)}(tg_{z_0})\,\mathrm{d}t\right)e^{-4\pi\lambda LM\mathrm{Tr}(S_d g_{z_0}{}^t g_{z_0})}\frac{\mathrm{d}z_0\,\mathrm{d}\lambda}{\lambda^4}.$$

5.4. **Waldspurger's formula for the toric period and subconvexity.** We begin by defining some purely local quantities. Let $p=\infty$ or let p be a prime dividing M. Let π_p be an irreducible, admissible unitary representation of $\mathrm{GL}_2(\mathbb{Q}_p)$. Fix some (unique up to multiples) invariant inner product $\langle\cdot,\cdot\rangle_p$ on π_p. If $p|M$, normalize the Haar measure on the subgroup $\mathbb{Q}_p^\times\backslash T_d(\mathbb{Q}_p)$ so that $\mathrm{vol}(\mathbb{Z}_p^\times\backslash T_d(\mathbb{Z}_p))=1$. Let v_p be a (unique up to multiples) spherical vector in the space of π_p and let $k_p^{(M)}$ be as defined in (5.42).

If $p|M$, define

$$J_p^{(M)}:=\int\limits_{\mathbb{Q}_p^\times\backslash T_d(\mathbb{Q}_p)}\frac{\langle tk_p^{(M)}v_p,k_p^{(M)}v_p\rangle_p}{\langle v_p,v_p\rangle_p}\,\mathrm{d}t.$$

If $p=\infty$, and $z_0\in\mathbb{H}$, define

$$J_\infty^{(z_0)}:=\frac{1}{\mathrm{vol}(T_d^1(\mathbb{R}))}\int\limits_{T_d^1(\mathbb{R})}\frac{\langle tg_{z_0}v_\infty,g_{z_0}v_\infty\rangle_\infty}{\langle v_\infty,v_\infty\rangle_\infty}\mathrm{d}t.$$

5.4.1. *The toric period in the case that g is a cusp form.* Let g be a Hecke–Maass cusp form and let $\pi=\otimes_v\pi_v$ be the irreducible, cuspidal automorphic representation of $\mathrm{GL}_2(\mathbb{A})$ generated by ϕ_g. Let $\langle g,g\rangle$ be the usual Petersson inner product. For each ϕ in the space of π, a famous formula of Waldspurger relates $|\left(\int_{\mathbb{A}^\times T_d(\mathbb{Q})\backslash T_d(\mathbb{A})}\phi(t)dt\right)|^2$ to the central value $L(1/2,\pi)L(1/2,\pi\otimes\chi_d)$ times some local factors.

We apply Waldspurger's formula [Wal85, Proposition 7] to the automorphic form

$$\phi(h)=\phi_g^{(M)}(hg_{z_0})=\phi_g(hg_{z_0}k^{(M)}).$$

This gives us the identity

$$\frac{\left|\int_{\mathbb{A}^\times T_d(\mathbb{Q})\backslash T_d(\mathbb{A})}\phi_g^{(M)}(tg_{z_0})\,\mathrm{d}t\right|^2}{\langle g,g\rangle} = C|d|^{-1/2}\frac{L^M(1/2,\pi)L^M(1/2,\pi\otimes\chi_d)}{L^M(1,\mathrm{ad}\,\pi)L^M(1,\chi_d)^2}J_\infty^{(z_0)}\prod_{p|M}J_p^{(M)}, \tag{5.44}$$

where $C>0$ is an absolute constant (to check that the constants appearing above are as required by Waldspurger's formula, we note [DPSS20, Sec. 3.4] that the constant C_T that relates the global Tamagawa measure and the product measure on $\mathbb{A}^\times T_d(\mathbb{Q})\backslash T_d(\mathbb{A})$ is given by $\frac{2w(K)}{h(d)\mathrm{vol}(T_d^1(\mathbb{R}))} = \frac{4\pi}{|d|^{1/2}\mathrm{vol}(T_d^1(\mathbb{R}))L(1,\chi_d)}$, where $w(K)$ is the number of roots of unity in K). Recall here that $L^M(1/2,\ldots)$ denotes the L-function with the factors at $p|M$ omitted, and changing these factors only has a mild effect (in particular at most M^ε) on the bounds we get for the central L-values.

We now claim that

$$|J_\infty^{(z_0)}|\ll 1, \quad \text{and} \tag{5.45}$$

$$J_p^{(M)}\ll_\varepsilon 1 \quad \text{for all } p|M. \tag{5.46}$$

To see (5.45) we just use the trivial bound $\left|\frac{\langle tg_{z_0}v_\infty, g_{z_0}v_\infty\rangle_\infty}{\langle v_\infty,v_\infty\rangle_\infty}\right|\le 1$. To see (5.46), note first that if p is inert or ramified in $\mathbb{Q}(\sqrt{d})$ then $\mathbb{Q}_p^\times\backslash T_d(\mathbb{Q}_p)$ is compact with volume $\asymp 1$, so $J_p^{(M)}\ll 1$ trivially in that case. Now, suppose that $p|M$ splits in $\mathbb{Q}(\sqrt{d})$. For brevity put $\Phi_p(g):=\frac{\langle gv_p,v_p\rangle_p}{\langle v_p,v_p\rangle_p}$ and let m be the highest power of p dividing M. Note that the function on $T_d(\mathbb{Q}_p)$ given by $t\mapsto\Phi_p((k_p^{(M)})^{-1}tk_p^{(M)}) = \frac{\langle tk_p^{(M)}v_p,k_p^{(M)}v_p\rangle_p}{\langle v_p,v_p\rangle_p}$ is $U_{T_d,p}(m)$-invariant. So using the definition of $J_p^{(M)}$, we see that $J_p^{(M)}\asymp p^{-m}\sum_{r\in\mathbb{Q}_p^\times U_{T_d,p}(m)\backslash T_d(\mathbb{Q}_p)}\Phi_p((k_p^{(M)})^{-1}rk_p^{(M)})$. For a set of representatives of $\mathbb{Q}_p^\times U_{T_d,p}(m)\backslash T_d(\mathbb{Q}_p)$ we can take the set described in Lemma 2.3 of [CMBS23]. Using Macdonald's formula [Bum97, Thm. 4.6.6] for Φ_p and bounding trivially, we obtain (5.46).

Combining (5.44), (5.45), (5.46) with the subconvexity bound for $L(1/2,\pi\otimes\chi_d)$ due to Petrow and Young [PY20] and the bound $L(1,\chi_d)\gg_\varepsilon d^{-\varepsilon}$ we deduce that

$$\left|\int_{\mathbb{A}^\times T_d(\mathbb{Q})\backslash T_d(\mathbb{A})}\phi_g^{(M)}(tg_{z_0})dt\right|^2\ll_{g,\varepsilon} M^\varepsilon|d|^{-\frac{1}{6}+\varepsilon}.$$

Therefore, combining the above with Proposition 5.4 we have

$$\begin{aligned}
G(D;L;g,\kappa) &\ll_{g,\varepsilon} |DL^2|^{k-\frac{3}{2}+\varepsilon}|d|^{-\frac{1}{6}}\frac{\mathrm{vol}(T_d^1(\mathbb{R}))}{2}\int_{\mathbb{R}^+}\lambda^{2k}|\kappa(\lambda)|\int_{T_d^1(\mathbb{R})\backslash\mathbb{H}}e^{-4\pi\lambda LM\mathrm{Tr}(S_dg_{z_0}{}^tg_{z_0})}\frac{\mathrm{d}z_0\,\mathrm{d}\lambda}{\lambda^4}\\
&= |DL^2|^{k-\frac{3}{2}+\varepsilon}|d|^{-\frac{1}{6}}\frac{1}{2}\int_{\mathbb{R}^+}\lambda^{2k}|\kappa(\lambda)|\int_{T_d^1(\mathbb{R})}1\int_{T_d^1(\mathbb{R})\backslash\mathbb{H}}e^{-4\pi\lambda LM\mathrm{Tr}(S_dg_{z_0}{}^tg_{z_0})}\frac{\mathrm{d}z_0\,\mathrm{d}\lambda}{\lambda^4}\\
&\ll_\kappa |DL^2|^{k-\frac{3}{2}+\varepsilon}|d|^{-\frac{1}{12}}\int_{\mathbb{R}^+}\lambda^{2k}\int_{\mathbb{H}}e^{-4\pi\lambda LM\mathrm{Tr}(S_dg_{z_0}{}^tg_{z_0})}\frac{\mathrm{d}z_0\,\mathrm{d}\lambda}{\lambda^4}\\
&\asymp 4^{-k}\pi^{-2k}\Gamma(k-3/2)\Gamma(k-2)|DL^2|^\varepsilon|d|^{-\frac{1}{12}},
\end{aligned} \tag{5.47}$$

where in the last step we have used the formula for Siegel's generalized Gamma function [Sie35, Hilfssatz 3].

5.4.2. *The toric period in the case that g is a Eisenstein series.* Recall that the Eisenstein series $E(s,z)$ is defined by

$$E(s,z) := \sum_{\gamma \in \left\{ \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \right\} \backslash \mathrm{SL}_2(\mathbb{Z})} \mathrm{Im}(\gamma z)^s$$

for $\mathrm{Re}(s) > 1$ and by meromorphic continuation to the rest of the complex plane. If $\mathrm{Re}(s) = 1/2$, these are called the unitary Eisenstein series. For every s away from the poles of the Eisenstein series, the function $E(s,\cdot)$ is a slowly growing function on $\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}$, and its adelization $\mathbb{E}(s,\cdot)$ is given by

$$\mathbb{E}(s,h) := \sum_{\gamma \in P_{\mathrm{GL}_2}(\mathbb{Q})\backslash \mathrm{GL}_2(\mathbb{Q})} f_\phi(s, \gamma h),$$

where P_{GL_2} is the usual parabolic subgroup of GL_2 and $f_\phi(s,\cdot) : \mathrm{GL}_2(\mathbb{A}) \longrightarrow \mathbb{C}^\times$ is the unique function satisfying $f_\phi(s, pk) = |a/b|_{\mathbb{A}}^s$ for all $p = \begin{pmatrix} a & * \\ & b \end{pmatrix} \in P_{\mathrm{GL}_2}(\mathbb{A})$, $k \in \mathrm{SO}(2) \prod_{p<\infty} \mathrm{GL}_2(\mathbb{Z}_p)$.

We now consider the inner integral in the expression given by Proposition 5.4 when $g = E(1/2 + ir, z)$ for some $r \in \mathbb{R}$. By a standard unfolding argument and bounds on local Tate integrals at infinity and the primes dividing M (see the proof of Prop. 12.5 of [BBK22]) we obtain that

$$\left| \int_{\mathbb{A}^\times T_d(\mathbb{Q})\backslash T_d(\mathbb{A})} \mathbb{E}(1/2 + ir, t g_{z_0} k^{(M)}) \, \mathrm{d}r \right| \ll_{r,\varepsilon} |d|^{-1/4} M^\varepsilon \frac{|L^M(1/2 + ir, \chi_d)|}{L^M(1, \chi_d)},$$

where the dependance of the constant on r is polynomial. Using subconvex bounds [PY23] on $L(1/2 + ir, \chi_d)$, we arrive at

$$\left| \int_{\mathbb{A}^\times T_d(\mathbb{Q})\backslash T_d(\mathbb{A})} \mathbb{E}\left(1/2 + ir, t g_{z_0} k^{(M)}\right) \, \mathrm{d}r \right| \ll_{r,\varepsilon} |d|^{-1/12+\varepsilon} M^\varepsilon. \tag{5.48}$$

Combining (5.48) with Proposition 5.4 and arguing as in (5.47) gives us

$$G(D; L; E(1/2 + ir, \cdot), \kappa) \ll_{r,\varepsilon,\kappa} 4^{-k} \pi^{-2k} \Gamma(k - 3/2) \Gamma(k-2) |DL^2|^\varepsilon |d|^{-\frac{1}{12}}.$$

5.4.3. *Conclusion.* Recall the definition of c_k from (3.6). We may summarize our results proved above as follows: that for g equal to either a Hecke–Maass cusp form or a unitary Eisenstein series, we have

$$G(D; L; g, \kappa) \ll_{g,\kappa,\varepsilon} k^{-3} |DL^2|^\varepsilon |d|^{-\frac{1}{12}} c_k \tag{5.49}$$

and the dependance is polynomial in r if $g = E(1/2 + ir, \cdot)$. Note that we have proved the bound (5.49) unconditionally; in particular, we did not assume GRH or GLH for the proof of (5.49).

5.5. **The endgame.** First we briefly recall the definition and basic properties of the incomplete Eisenstein series. For each $\Psi \in C_c^\infty(\mathbb{R}^+)$ the incomplete Eisenstein series is defined by

$$E(\Psi, z) := \sum_{\gamma \in \left\{ \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \right\} \backslash \mathrm{SL}_2(\mathbb{Z})} \Psi(\mathrm{Im}(\gamma z)). \tag{5.50}$$

By Mellin inversion and Cauchy's theorem, we have

$$E(\Psi, z) = \frac{\widetilde{\Psi}(1)}{\mathrm{vol}(\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H})} + \int_{-\infty}^{\infty} \widetilde{\Psi}(1/2 + it) E(1/2 + it, z) \, \frac{\mathrm{d}t}{2\pi i}. \tag{5.51}$$

Let $g = E(\Psi, \cdot)$ be an incomplete Eisenstein series. Then we have

$$\int\limits_{\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}} g(u+iy)\frac{\mathrm{d}u\,\mathrm{d}y}{y^2} = \widetilde{\Psi}(1).$$

We now complete the proof of Proposition 3.2. We need to show that (3.9) holds for each fixed $g \in C_c^\infty(\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H})$ and $\kappa \in C_c^\infty(\mathbb{R}^+)$. In the next lemma we reduce to the case that g is a Hecke–Maass cusp form or an incomplete Eisenstein series.

Lemma 5.5. *Let $F \in S_k(\Gamma)$ traverse a family of Hecke eigenforms that are Saito–Kurokawa lifts and let $\kappa \in C_c^\infty(\mathbb{R}^+)$ be fixed. Suppose that for each fixed g that is equal to either a Hecke–Maass cusp form or an incomplete Eisenstein series on $\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}$, the limit* (3.9) *is true. Then* (3.9) *is true for each fixed $g \in C_c^\infty(\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H})$.*

Proof. Let $g \in C_c^\infty(\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H})$ and let $\varepsilon > 0$. It is known that the class $C_c^\infty(\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H})$ is contained in the uniform span of the Hecke–Maass cusp forms and incomplete Eisenstein series (see [Iwa02]). So we can find a finite set of $g_i \in C_c^\infty(\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}), 1 \le i \le r$ each of which is either a Hecke–Maass cusp form or an incomplete Eisenstein series such that

$$\left\| g - \sum_{i=1}^{r} g_i \right\|_\infty < \varepsilon. \tag{5.52}$$

For brevity, put $g_0 = \sum_{i=1}^r g_i$. Since (3.9) holds for each g_i by assumption, it follows that

$$\left| \frac{1}{\|F\|_2^2} \sum_{T \in \Lambda_2/\mathrm{SL}_2(\mathbb{Z})} \frac{|R(T)|^2}{\varepsilon(T)} |\mathrm{disc}(T)|^{k-\frac{3}{2}} G(T; g_0, \kappa) - \frac{\widetilde{\kappa}(3)}{2\mathrm{vol}(\Gamma\backslash\mathbb{H}_2)} \int\limits_{\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}} g_0(u+iy)\frac{\mathrm{d}u\,\mathrm{d}y}{y^2} \right| < \varepsilon \tag{5.53}$$

for sufficiently large k. By combining (5.52) and (5.53), and recalling (5.17), it follows that for all sufficiently large k we have

$$\begin{aligned} &\left| \frac{1}{\|F\|_2^2} \sum_{T \in \Lambda_2/\mathrm{SL}_2(\mathbb{Z})} \frac{|R(T)|^2}{\varepsilon(T)} |\mathrm{disc}(T)|^{k-\frac{3}{2}} G(T; g, \kappa) - \frac{\widetilde{\kappa}(3)}{2\mathrm{vol}(\Gamma\backslash\mathbb{H}_2)} \int\limits_{\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}} g(u+iy)\frac{\mathrm{d}u\,\mathrm{d}y}{y^2} \right| \\ &< \varepsilon \left(1 + \left| \frac{1}{\|F\|_2^2} \sum_{T \in \Lambda_2/\mathrm{SL}_2(\mathbb{Z})} \frac{|R(T)|^2}{\varepsilon(T)} |\mathrm{disc}(T)|^{k-\frac{3}{2}} G(T; 1, |\kappa|) \right| + \left| \frac{\widetilde{\kappa}(3)\mathrm{vol}(\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H})}{2\mathrm{vol}(\Gamma\backslash\mathbb{H}_2)} \right| \right) \\ &\ll_\kappa \varepsilon. \end{aligned}$$

By taking ε arbitrarily small, the proof of the lemma is complete. □

So to finish the proof of Proposition 3.2 we need to show that for each fixed $\kappa \in C_c^\infty(\mathbb{R}^+)$ and each fixed g equal to either a Hecke–Maass cusp form or an incomplete Eisenstein series, the limit (3.9) holds. Moreover, the argument leading up to (5.19) shows that in the left hand side of (3.9) we may restrict to the terms corresponding to $\mathrm{disc}(T) \asymp_\kappa k^2$; we are implicitly using here that g is a fixed bounded function and so we may write $g(z) \ll_g 1$.

We first consider the case that g is a Hecke–Maass cusp form where we need to show that (see (5.34))

$$\frac{1}{\|F\|_2^2} \sum_{\substack{L, D \\ 0 > D \equiv 0,1 (4) \\ DL^2 \asymp_\kappa k^2}} h(D)|R(D;L)|^2 G(D; L; g, \kappa) \longrightarrow 0 \tag{5.54}$$

as $k \longrightarrow \infty$. Using (5.49) we obtain

$$\begin{aligned}
&\frac{1}{\|F\|_2^2} \sum_{\substack{L,D \\ 0>D\equiv 0,1(4) \\ DL^2 \asymp_\kappa k^2}} h(D)|R(D;L)|^2 |G(D;L;g,\kappa)| \\
&\ll_{g,\kappa,\varepsilon} k^{-3} \frac{c_k}{\|F\|_2^2} \sum_{\substack{L,M,d \\ d\in\mathcal{D} \\ dL^2M^2 \asymp_\kappa k^2}} (|d|LM)^\varepsilon |d|^{-\frac{1}{12}} h(dM^2)|R(dM^2;L)|^2 \\
&= k^{-3} \frac{c_k}{\|F\|_2^2} \sum_{d\in\mathcal{D}} |d|^{-\frac{1}{12}} \sum_{\substack{L,M \\ L^2M^2 \asymp_\kappa \frac{k^2}{|d|}}} (LM)^\varepsilon h(dM^2)|R(dM^2;L)|^2.
\end{aligned}$$

By Lemma 4.13, we have under GLH for each positive integer N,

$$\frac{c_k}{\|F\|_2^2} \sum_{LM=N} (LM)^\varepsilon h(dM^2)|R(dM^2;L)|^2 \ll_\varepsilon (k|d|N)^\varepsilon |d|^{1/2} N.$$

So

$$\begin{aligned}
\frac{1}{\|F\|_2^2} \sum_{\substack{L,D \\ 0>D\equiv 0,1(4) \\ DL^2 \asymp_\kappa k^2}} h(D)|R(D;L)|^2 G(D;L;g,\kappa) &\ll_{g,\kappa,\varepsilon} k^{-3+\varepsilon} \sum_{\substack{N,d \\ d\in\mathcal{D} \\ |d|N^2 \asymp_\kappa k^2}} (|d|N^2)^{1/2+\varepsilon} |d|^{-\frac{1}{12}} \\
&\ll_\varepsilon k^{-1/6+\varepsilon},
\end{aligned}$$

which completes the proof of (5.54).

We next consider the case that $g = E(\Psi, \cdot)$ is an incomplete Eisenstein series. By (5.34), in this case we need to show that

$$\frac{1}{\|F\|_2^2} \sum_{L,D} h(D) \frac{|R(D;L)|^2}{\varepsilon(D)} G(D;L;g,\kappa) - \frac{\widetilde{\Psi}(1)}{2 \cdot \mathrm{vol}(\Gamma\backslash\mathbb{H}_2)} \widetilde{\kappa}(3) \longrightarrow 0 \tag{5.55}$$

as $k \longrightarrow \infty$. Using (5.51), we can write the expression above as $L_1 + L_2$, where

$$L_1 := \frac{\widetilde{\Psi}(1)}{\mathrm{vol}(\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H})\|F\|_2^2} \sum_{L,D} h(D) \frac{|R(D;L)|^2}{\varepsilon(D)} G(D;L;1,\kappa) - \frac{\widetilde{\Psi}(1)}{2 \cdot \mathrm{vol}(\Gamma\backslash\mathbb{H}_2)} \widetilde{\kappa}(3)$$

and

$$L_2 := \int_{-\infty}^{\infty} \widetilde{\Psi}(1/2+it) \frac{1}{\|F\|_2^2} \sum_{L,D} h(D) \frac{|R(D;L)|^2}{\varepsilon(D)} G(D;L;E(1/2+it,z),\kappa) \frac{\mathrm{d}t}{2\pi i}$$

By (5.17), which treated the case $g = 1$, we have $L_1 \longrightarrow 0$ as $k \longrightarrow \infty$. On the other hand, using (5.49) and following an identical argument to the cusp form case treated above, we get that $L_2 \longrightarrow 0$ as $k \longrightarrow \infty$. This completes the proof of (5.55).

The proof of Theorem 3.2 is complete.

6. Equidistribution of zero divisors

One consequence of the mass equidistribution for classical holomorphic modular forms is that the zeros of such forms become equidistributed with respect to hyperbolic measure as the weight tends to infinity. This has been proved by Shiffman and Zelditch [SZ99] for compact hyperbolic surfaces and extended to the non-compact case of the modular surface by Rudnick [Rud05]. Methods of these papers have also been applied by Marshall [Mar11] to show the analogous statement

about the equidistribution of the smooth parts of zero divisors of holomorphic modular forms of cohomological type on GL_2. As an application of our mass equidistribution result, we will derive a similar equidistribution result for Saito-Kurokawa lifts under GRH. The method of proof closely follows the previous works, but we shall provide a self-contained proof for the sake of completeness as the set-up is slightly different compared to the aforementioned papers.

To put our result into the context of [SZ99], it is well-known that $Y_2 = \Gamma\backslash\mathbb{H}_2$ is the moduli space of principally polarized abelian varieties of dimension two and that it carries a universal principally polarized abelian variety $\pi : \mathcal{X}_2 \longrightarrow Y_2$. This provides Y_2 with a natural vector bundle, called the Hodge bundle, defined as

$$E := \pi_* \left(\Omega^1_{\mathcal{X}_2/Y_2} \right),$$

where π_* is the pushforward and $\Omega^1_{\mathcal{X}_2/Y_2}$ is the sheaf of relative differentials. Each irreducible representation ρ of the Levi subgroup GL_2 of GSp_4 equips Y_2 with a new vector bundle[7] E_ρ by applying ρ to the transition maps of E. In particular, $\rho = \det$ gives the determinant bundle denoted by L. It is well-known that classical Siegel modular forms of weight k and full level for the group GSp_4 are sections of $L^{\otimes k}$.

Let Z_F be the zero divisor of a holomorphic function F on Y_2, that is,

$$Z_F := \sum_i \mathrm{ord}_{V_i}(F) V_i,$$

where V_i are the irreducible subvarieties of $F^{-1}(0)$ and $\mathrm{ord}_{V_i}(F)$ is the order of vanishing of F on V_i. The zero divisor defines a distribution, called the current of integration, on the space of smooth compactly supported differential forms on Y_2 via

$$[Z_F] : \quad \eta \mapsto \int_{Z_F} \eta := \sum_i \mathrm{ord}_{V_i}(F) \int_{V_i} \eta.$$

As a consequence of the mass equidistribution we show the equidistribution of zero divisors on the Siegel modular variety, which may be interpreted as saying that for Saito-Kurokawa lifts the subvarieties V_i become equidistributed as Lelong $(2,2)$-currents (or more simply as measures of integration) with respect to the induced Kähler form ω on Y_2 under GRH as the weight tends to infinity.

The proof utilizes basic compactness properties of plurisubharmonic functions, which are collected in the following two lemmas.

Lemma 6.1. *Let $\{u_j\}$ be a family of plurisubharmonic functions on Ω which are locally uniformly bounded from above. Then either*

i) *$u_j \longrightarrow -\infty$ uniformly on compact sets*
or
ii) *There exists a subsequence $\{u_{j_k}\}$ such that $u_{j_k} \longrightarrow u$ for some plurisubharmonic function u. In this case $\limsup_{j\longrightarrow\infty} u_j \leq u$ and $\limsup_{j\longrightarrow\infty} u_j = u$ almost everywhere.*

Lemma 6.2 (Hartog's lemma)**.** *If $\{u_j\}$ is a family of plurisubharmonic functions on Ω which are locally uniformly bounded from above and there exists a continuous map $\varphi : \Omega \longrightarrow \mathbb{C}$ so that $\limsup_{j\longrightarrow\infty} u_j \leq \varphi$, then $\max(u_j, \varphi) \longrightarrow \varphi$ locally uniformly on Ω.*

Both of these results can be proven verbatim as their counterparts for subharmonic functions. For these see [Hö03, Theorem 4.1.9] and [Ran95, Theorem 3.4.3], respectively. For the second statement, see also [Kli91, Theorem 2.9.14 (ii)].

[7]The Hodge bundle corresponds to the standard representation.

We also need the Poincaré-Lelong formula from complex analytic geometry [Dem, Chapter 2], which is formulated in a special case below.

Lemma 6.3 (Poincaré-Lelong formula)**.** *For a holomorphic function F on $\mathbb{H}_2$ we have the equality*

$$\frac{i}{\pi}\log(|F|)\partial\overline{\partial} = [Z_F]$$

as currents of integration of bidegree $(2,2)$*.*

Now we have all the necessary tools to prove Theorem 1.3.

Proof of Theorem 1.3. By standard approximation argument it suffices to prove the statement for η replaced by its symmetrized form

$$F_\eta := \sum_{\gamma\in\Gamma}\gamma^*\eta$$

By unfolding we have

$$\int\limits_{Z_{F_k}} F_\eta = \int\limits_{\widetilde{Z_{F_k}}} \eta,$$

where $\widetilde{Z_{F_k}}$ is the zero divisor of Γ-periodic extension of F_k to $\mathbb{H}_2$. By Lemma 6.3 we have

$$\begin{aligned}\int\limits_{\widetilde{Z_{F_k}}} \eta &= \frac{i}{\pi}\int\limits_{\mathbb{H}_2}\log(|F_k|)\partial\overline{\partial}\eta \\ &= -\frac{i}{\pi}\int\limits_{\mathbb{H}_2}\log\left((\det Y)^{k/2}\right)\partial\overline{\partial}\eta + \frac{i}{\pi}\int\limits_{\mathbb{H}_2}\log\left((\det Y)^{k/2}|F_k|\right)\partial\overline{\partial}\eta.\end{aligned}$$

Integrating by parts and refolding the first term on the right-hand side is

$$-\frac{k}{2}\cdot\frac{i}{\pi}\int\limits_{\mathbb{H}_2}\eta\partial\overline{\partial}\log\left((\det Y)\right) = k\int\limits_{\mathbb{H}_2}\omega\wedge\eta = k\int\limits_{\Gamma\backslash\mathbb{H}_2}\omega\wedge F_\eta.$$

Combining these computations yields

$$\frac{1}{k}\int\limits_{Z_{F_k}} F_\eta = \int\limits_{\Gamma\backslash\mathbb{H}_2}\omega\wedge F_\eta + \frac{i}{\pi k}\int\limits_{\mathbb{H}_2}\log\left((\det Y)^{k/2}|F_k|\right)\partial\overline{\partial}\eta.$$

Hence it suffices to show that

$$\frac{1}{k}\int\limits_{\mathbb{H}_2}\log\left((\det Y)^{k/2}|F_k|\right)\partial\overline{\partial}\eta \longrightarrow 0$$

as $k\longrightarrow\infty$, or equivalently

$$\frac{1}{k}\int\limits_{\mathbb{H}_2}\log(|F_k|)\partial\overline{\partial}\eta \longrightarrow -\frac{1}{2}\int\limits_{\mathbb{H}_2}\log(\det Y)\partial\overline{\partial}\eta. \tag{6.1}$$

Suppose otherwise: there exists some smooth compactly supported differential form η_0 of bidegree $(2,2)$ on $\Gamma\backslash\mathbb{H}_2$ and a sequence of Saito-Kurokawa lifts $\{F_k\}$ so that (6.1) does not hold. We make two crucial observations:

i) Functions $\frac{1}{k}\log(|F_k|)$ are plurisubharmonic on $\Gamma\backslash\mathbb{H}_2$.

ii) $\limsup_{k\longrightarrow\infty}\frac{1}{k}\log(|F_k|) \leq -\frac{1}{2}\log(\det Y)$ locally uniformly.

The first observation is a well-known fact as the forms F_k are holomorphic. For the second one, Blomer [Blo15] has shown that $\|(\det Y)^{k/2}F_k\|_\infty \ll_\varepsilon k^{3/4+\varepsilon}$ uniformly on compact sets for Saito-Kurokawa lifts F_k under GRH. Thus

$$\frac{1}{k}\log(|F_k|) \leq -\frac{1}{2}\log(\det Y) + \frac{\left(\frac{3}{4}+\varepsilon\right)\log k + O(1)}{k},$$

which gives the desired estimate.

The conclusion is that $\{\frac{1}{k}\log(|F_k|)\}$ is a family of plurisubharmonic functions, which are locally uniformly bounded from above. Thus Lemma 6.1 tells that either $\frac{1}{k}\log(|F_k|) \longrightarrow -\infty$ uniformly on compact sets or that there exists a subsequence of $\{\frac{1}{k}\log(|F_k|)\}$ converging to some plurisubharmonic function. We will derive a contradiction in both cases.

Case 1. Suppose that $\frac{1}{k}\log(|F_k|) \longrightarrow -\infty$ uniformly on compact sets. Then in particular $\frac{1}{k}\log(|F_k|) \longrightarrow -\infty$ uniformly on the support of η_0. Hence, there exists $K > 0$ so that for $k \geq K$ and $Z \in \operatorname{supp}\eta_0$ we have $\frac{1}{k}\log(|F_k(Z)|) \leq -H$, where $H := \max\{\det Y \,:\, Z \in \operatorname{supp}\eta_0\}$, or equivalently $|F_k(Z)|^2 \leq e^{-2kH}$. This means that for all smooth differential forms η of bidegree $(2,2)$ with $\operatorname{supp}\eta \subset \operatorname{supp}\eta_0$ we have

$$\int\limits_{Y_2} |F_k|^2(\det Y)^k\omega\wedge\eta \longrightarrow 0$$

as $k \longrightarrow \infty$, which is impossible by the mass equidistribution.

Case 2. Suppose that $\frac{1}{k}\log(|F_k|) \longrightarrow u$ for some plurisubharmonic function u along a subsequence which is still denote by $\{\frac{1}{k}\log(|F_k|)\}$. We know that $\limsup_{k\longrightarrow\infty}\frac{1}{k}\log(|F_k|) \leq u$ and $\limsup_{k\longrightarrow\infty}\frac{1}{k}\log(|F_k|) = u$ almost everywhere. From ii) we have $u(Z) \leq -\frac{1}{2}\log(\det Y)$ almost everywhere. From our counter-assumption to (6.1) we have $u(Z) \neq -\frac{1}{2}\log(\det Y)$ in a set of positive measure. Thus, there exists $\delta > 0$ so that $u(Z) < -\frac{1}{2}\log(\det Y) - \delta$ on some compact open subset U. By Lemma 6.2 there exists $K = K(\delta, U)$ so that for all $k \geq K$ we have $\frac{1}{k}\log(|F_k(Z)|) < -\frac{1}{2}\log(\det Y) - \delta/2$ on U and consequently $(\det Y)^k|F_k(Z)|^2 \leq e^{-k\delta}$ on U. This obviously contradicts the mass equidistribution as in the previous case.

We conclude that (6.1) holds and the proof is completed. □

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