ON THE REAL ZEROES OF HALF-INTEGRAL WEIGHT HECKE CUSP FORMS

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ABSTRACT. We examine the distribution of zeroes of half-integral weight Hecke cusp forms on the manifold $\Gamma_0(4)\backslash\mathbb{H}$ near a cusp at infinity. In analogue of the Ghosh-Sarnak conjecture for classical holomorphic Hecke cusp forms, one expects that almost all of the zeroes sufficiently close to this cusp lie on two vertical geodesics $\mathrm{Re}(s) = -1/2$ and $\mathrm{Re}(s) = 0$ as the weight tends to infinity. We show that for $\gg K^2/(\log K)^{3/2}$ of the half-integral weight Hecke cusp forms in the Kohnen plus subspace with weight bounded by a large constant K, the number of such "real" zeroes grows almost at the expected rate. We also obtain a weaker lower bound for the number of real zeroes that holds for a positive proportion of forms. One of the key ingredients is the asymptotic evaluation of averaged first and second moments of quadratic twists of modular L-functions.

1. Introduction

Studying the distribution of zeroes of automorphic forms has attracted attention both historically and more recently. A classical result in the theory of holomorphic modular forms for the group $\mathrm{SL}_2(\mathbb{Z})$ on the upper half $\mathbb{H}$ of the complex plane is the so-called valence formula. It states that the number of (properly weighted) zeroes of such form f with weight k is asymptotically $k/12$ inside the fundamental domain $\mathcal{D} := \mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H} = \{z \in \mathbb{H} : -1/2 \leq \mathrm{Re}(z) \leq 1/2, |z| \geq 1\}$. Interestingly, it turns out that on smaller scales the distribution of zeroes of different types of automorphic forms can vary drastically. For instance, Rankin and Swinnerton-Dyer [42] have shown that all the zeroes of holomorphic Eisenstein series lie on the arc $\{|z| = 1\}$ and moreover these zeroes are uniformly distributed there as $k \longrightarrow \infty$. In contrast, powers of the modular discriminant have a single zero of high multiplicity at infinity. Another type of behaviour is displayed by Hecke cusp forms, which form a natural basis for the space of modular forms of a given weight.

For the latter forms Rudnick [43] showed that the equidistribution of zeroes inside $\mathcal{D}$, as the weight of the form tends to infinity, follows from the Quantum Unique Ergodicity conjecture (QUE) of himself and Sarnak [44]. As the QUE conjecture has since been shown to hold by Holowinsky and Soundararajan [15], Rudnick's equidistribution result is unconditional. Given this, it is natural to wonder about the finer distributional behaviour of the zeroes of Hecke cusp forms, e.g. their distribution within subsets of the fundamental domain $\mathcal{D}$ that shrink as the weight grows.

Such questions were first explored by Ghosh and Sarnak [13] by considering the distribution of zeroes in shrinking domains around the cusp of the modular surface $\mathcal{D}$ at infinity. To be precise, they studied zeroes inside the Siegel sets $\mathcal{D}_Y := \{z \in \mathcal{D} : \mathrm{Im}(z) \geq Y\}$ with $Y \longrightarrow \infty$ sufficiently fast along with $k \longrightarrow \infty$. This can be regarded as a shrinking ball around the cusp at infinity as the hyperbolic area of $\mathcal{D}_Y$ equals $1/Y$ and so tends to zero as the weight tends to infinity. Ghosh and Sarnak observed that, although the number of zeroes inside $\mathcal{D}_Y$ is proportional to the area of the domain, the statistical behaviour of the zeroes was very different from the uniform distribution when $\sqrt{k \log k} \ll Y \ll k$. Indeed, they observed that equidistribution of zeroes should not happen inside these sets and, based on numerical evidence and a random model, were lead to conjecture that almost all of the zeroes inside $\mathcal{D}_Y$, for Y in the same range as before, concentrate on the half-lines $\mathrm{Re}(s) = -1/2$ and $\mathrm{Re}(s) = 0$. They termed such zeroes to be "real" as the cusp form is itself real-valued on these lines[1]. Ghosh and Sarnak conjectured that 50% of the zeroes in these shrinking Siegel sets should lie on the line $\mathrm{Re}(s) = -1/2$ and likewise 50% on the line $\mathrm{Re}(s) = 0$. Furthermore, they obtained some results in this direction. In the end they were able to produce $\gg_\varepsilon (k/Y)^{1/2-1/40-\varepsilon}$ such real zeroes in the range $\sqrt{k \log k} \ll Y < k/100$. The exponent was later improved to $1/2 - \varepsilon$ by Matomäki [36]. Producing real zeroes on the individual lines $\mathrm{Re}(s) = -1/2$ and $\mathrm{Re}(s) = 0$ is more challenging, but Ghosh

[1]In the same paper they also considered zeroes on the arc $\{|z| = 1\}$ where $z^{k/2}f(z)$ is real-valued.

and Sarnak succeeded, the current best results, which are of polynomial growth, are again due to Matomäki [36]. Related to this Matomäki, Lester, and Radziwiłł [31] have also obtained some further results that we shall discuss below in more detail.

One may of course speculate that similar phenomenon holds also for other types of automorphic forms. The analogue of the QUE conjecture is known for half-integral weight Hecke cusp forms thanks to the work of Lester and Radziwiłł [32] under the Generalised Riemann Hypothesis (GRH). Similarly to Rudnick's work this implies the equidistribution of zeroes inside the fundamental domain $\Gamma_0(4)\backslash\mathbb{H}$, where $\Gamma_0(4)$ is the Hecke congruence subgroup of level 4, of course conditionally on GRH. Given this, it is natural to wonder whether the aforementioned results concerning the small scale distribution of zeroes generalise to these forms. The goal of the present paper is to address this as it seems that these questions have not been explored previously for the half-integral weight Hecke cusp forms.

This is not that surprising as the half-integral weight situation has features that are not present in the setting of classical holomorphic cusp forms. Indeed, Ghosh and Sarnak (and subsequent works) exhibited a relationship between real zeroes of integral weight Hecke cusp forms and sign changes of their Fourier coefficients. We shall do the same in setting of half-integral weight Hecke cusp forms, but here there are certain key differences that make adapting the methods used in the integral weight setting challenging. In the classical case the methods of [13, 36, 31] rely in a fundamental way on the multiplicativity of the Fourier coefficients of Hecke cusp forms. However, the Fourier coefficients of half-integral weight Hecke cusp forms lack this property, except at squares. Because of this the methods used in the previous works are not directly applicable in our setting and we have to use different tools to investigate the distribution of real zeroes.

The fundamental domain $\mathcal{F} := \Gamma_0(4)\backslash\mathbb{H}$ is taken to be the domain in the upper half-plane bordered by the vertical lines $\sigma = \pm\frac{1}{2}$ and circles[2] $B(-\frac{1}{3}, \frac{1}{3})$, $B(\frac{1}{5}, \frac{1}{5})$, and $B(\frac{3}{8}, \frac{1}{8})$. Notice that this fundamental domain has three cusps at ∞, 0, and $\frac{1}{2}$ (these have widths 1, 4, and 1, respectively).

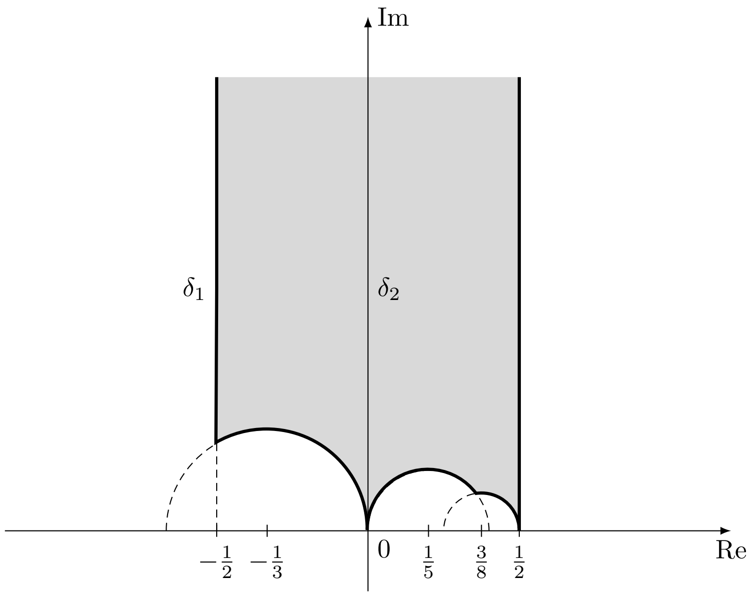

Figure 1. Fundamental domain for the action of the group $\Gamma_0(4)$ on $\mathbb{H}$.

It is well-known that any half-integral weight Hecke cusp form may be normalised to have real Fourier coefficients[3] $c_g(n)$ and we impose this normalisation throughout the paper.

[2]Here $B(x, r)$ denotes the circle with radius r centred at x.

[3]Indeed, the numbers $c_g(n)n^{k+\frac{1}{2}}$ lie in the field generated over $\mathbb{Q}$ by the Fourier coefficients of its Shimura lift, which is a level 1 Hecke eigenform of weight $2k$, so these numbers are real and algebraic (see [29, Proposition 4.2] and also the remarks before [26, Theorem 1]).

Analogous to the integral weight setting, we study the real zeroes, that is zeroes on the two geodesic segments

$$\delta_1 := \left\{ s \in \mathbb{C} : \operatorname{Re}(s) = -\frac{1}{2} \right\} \qquad \text{and} \qquad \delta_2 := \{ s \in \mathbb{C} : \operatorname{Re}(s) = 0 \}$$

on which the cusp form takes real values in the normalisation above. Our first main result establishes that there are almost the expected amount (our result is optimal up to a power of the logarithm) of real zeroes for many half-integral weight Hecke cusp forms. Throughout the article, let k be a positive integer. We write $S_{k+\frac{1}{2}}(4)$ for the space of half-integral weight cusp forms of weight $k+\frac{1}{2}$ and level 4. Also we denote $S^+_{k+\frac{1}{2}}(4) \subset S_{k+\frac{1}{2}}(4)$ for the Kohnen plus subspace and let $B^+_{k+\frac{1}{2}}$ denote a fixed Hecke eigenbasis for $S^+_{k+\frac{1}{2}}(4)$. Note that for half-integral weight cusp forms we cannot normalise the coefficient $c_g(1)$ to be equal to one without losing the algebraicity of the Fourier coefficients. This means that, unlike in the integral weight case, there is no canonical choice for $B^+_{k+\frac{1}{2}}$, but this causes no problems for us. We write $\mathcal{Z}(g) := \{z \in \mathcal{F} : g(z) = 0\}$ for the set of zeroes of $g \in S^+_{k+\frac{1}{2}}(4)$ and let $\mathcal{F}_Y := \{z \in \mathcal{F} : \operatorname{Im}(z) \geq Y\}$. Finally, set[4]

$$\mathcal{S}_K := \bigcup_{k \sim K} B^+_{k+\frac{1}{2}}$$

and note that the cardinality of this set is $\asymp K^2$. With these notations our first main result may be stated as follows.

Theorem 1.1. *Let K be a large parameter, and $j \in \{1,2\}$. Then for $\gg K^2/(\log K)^{3/2}$ of the forms $g \in \mathcal{S}_K$ we have*

$$\#\{\mathcal{Z}(g) \cap \delta_j \cap \mathcal{F}_Y\} \gg \frac{K}{Y} (\log K)^{-23/2}$$

for $\sqrt{K \log K} \leq Y \leq K^{1-\delta}$ with any small fixed constant $\delta > 0$.

Remark 1.2. Here we have not tried to optimise the powers of logarithm. The exponents present are conveniently chosen so to that the various exponents that appear in the proof are simple fractions.

As indicated above, the methods used in the integral weight case are hard to implement to our setting. However, building on the multiplicativity of the Fourier coefficients at squares, we show that for positive proportion of forms it also possible to get some real zeroes.

Theorem 1.3. *Let K be a large parameter, $\varepsilon > 0$ be an arbitrarily small fixed number, and $j \in \{1,2\}$. Then for a proportion $1/2 - \varepsilon$ of the forms $g \in \mathcal{S}_K$ we have*

$$\#\{\mathcal{Z}(g) \cap \delta_j \cap \mathcal{F}_Y\} \gg \left(\frac{K}{Y}\right)^{1/2} (\log K)^{-1}$$

for $\sqrt{K \log K} \leq Y \leq K^{1-\delta}$ with any small fixed constant $\delta > 0$.

As far as the author is aware of, these are the first results concerning the small scale distribution of zeroes of Hecke cusp forms in the half-integral weight setting[5]. Recall that the domain $\mathcal{F}_Y$ contains $\asymp k/Y$ zeroes of $g \in S^+_{k+\frac{1}{2}}(4)$, at least conditionally on GRH. The above results give progress towards the conjecture[6] that the number of real zeroes of $g \in S^+_{k+\frac{1}{2}}(4)$ is $\gg k/Y$ on $\delta_j \cap \mathcal{F}_Y$ for both of the individual lines δ_j in the same range of Y as before[7]. That is, apart from log-powers our first result shows that we get the expected number of real zeroes for many, albeit not a positive proportion of, forms. The above theorems may be compared

[4]Here, and throughout the paper, the notation $\ell \sim L$ means that $L \leq \ell \leq 2L$.

[5]However, for elements in certain canonical basis, see [12].

[6]In the integral weight case a probabilistic model for this is given by Ghosh and Sarnak [13, Section 6.] and the same model works also in the half-integral weight setting.

[7]One may refine this conjecture to state that each of the individual lines δ_j contain 50% of the zeroes.

to results of Lester, Matomäki, and Radziwiłł [31] who showed that in the setting of classical holomorphic Hecke cusp forms one has (for both $j \in \{1,2\}$)

$$\#\{\mathcal{Z}(f) \cap \delta_j \cap \mathcal{D}_Y\} \gg_\varepsilon \left(\frac{k}{Y}\right)^{1-\varepsilon}$$

for all $\varepsilon > 0$ under the Generalised Lindelöf Hypothesis, and unconditionally that for almost all forms one has

$$\#\left\{\mathcal{Z}(f) \cap \delta_j \cap \mathcal{D}_Y\right\} \asymp \frac{k}{Y}$$

in the same range of Y as above. However, as indicated above, our techniques used to prove Theorem 1.1 are very different compared to those used in the integral weight setting.

2. The Strategy

In this section we describe the main ideas that go into the proofs of the main theorems. Let k be a positive integer and let g be a Hecke cusp form of half-integral weight $k + \frac{1}{2}$, level 4, that belongs to Kohnen's plus subspace. Every such g has a Fourier expansion

$$g(z) = \sum_{\substack{n=1 \\ (-1)^k n \equiv 0,1 \pmod 4}}^{\infty} c_g(n) n^{\frac{k}{2}-\frac{1}{4}} e(nz) \tag{2.1}$$

with normalised real Fourier coefficients $c_g(n)$.

The Fourier coefficients $c_g(n)$ encode arithmetic information. For instance, Waldspurger's formula shows that for fundamental discriminants d with $(-1)^k d > 0$, $|c_g(|d|)|^2$ is proportional to the central value of an L-function, and so the magnitude of the L-function essentially determines the size of the coefficient $c_g(n)$. We shall use this fact repeatedly.

Let us then explain how to exploit the information about the Fourier coefficients $c_g(n)$ in order to produce real zeroes. By the steepest descent argument it turns out that on the half-lines $\sigma + iy$, with $\sigma \in \{-\frac{1}{2}, 0\}$ and $y > 0$, the values $g(\sigma + iy_\ell)$ for $y_\ell := (k-1/2)/4\pi\ell$ and $1 \ll \ell \ll k/Y$, are essentially determined by the single Fourier coefficient $c_g(\ell)$. Morally this reduces finding zeroes on these lines (i.e., real zeroes) to studying sign changes of the Fourier coefficients $c_g(n)$. The latter question has been considered by many authors following the works of Knopp-Kohnen-Pribitkin [24] and Bruinier-Kohnen [9], the former of which showed that such forms have infinitely many sign changes. Subsequent works [16, 25, 30] showed that the sequence $\{c_g(n)\}_n$ exhibits many sign changes under suitable conditions. It is essential for our approach to obtain quantitative results that are uniform in the weight aspect.

Due to the lack of multiplicativity of these coefficients, the methods used to produce sign changes [13, 36, 31] are not readily available in our setting. For some special subsequences of coefficients, methods of multiplicative number theory are useful. For example, for a fixed positive squarefree integer t the coefficients $c_g(tm^2)$ are multiplicative. This can be used in the proof of Theorem 1.3 as we now explain.

For this we rely on the following identity relating certain Fourier coefficients of half-integral weight cusp form and the Fourier coefficients of its Shimura lift at primes:

$$c_g(|d|p^2) = c_g(|d|)\left(\lambda_f(p) - \frac{\chi_d(p)}{\sqrt{p}}\right). \tag{2.2}$$

Here p is any prime, d is a fundamental discriminant with $(-1)^k d > 0$, and χ_d is the unique real quadratic character of conductor d.

To benefit from this we have the following auxiliary result concerning the non-vanishing of $c_g(|d|)$, which is somewhat stronger that we will actually require.

Proposition 2.1. *Let K be a large parameter. Then there exists a small absolute constant $\delta > 0$ so that for any odd fundamental discriminant d with $|d| \le K^\delta$ we have $c_g(|d|) \neq 0$ for a proportion $\ge 1/2 - \varepsilon$, with any fixed $\varepsilon > 0$, of $g \in \bigcup_{\substack{k \sim K \\ (-1)^k d > 0}} S^+_{k+\frac{1}{2}}(4)$ as $K \longrightarrow \infty$.*

This follows by combining asymptotics for the mollified second and fourth moments of the Fourier coefficients $c_g(|d|)$ using the Cauchy-Schwarz inequality and removing the harmonic weights using an approach of Iwaniec and Sarnak [19] detailed in [28, 2]. The relevant moments are connected to moments of L-functions via Waldspurger's formula and so the mollifier is constructed to counteract the large values of $L(1/2, f \otimes \chi_d)$. To be more specific, let us define the mollifier to be a short linear form

$$\mathcal{M}_{f,d} := \sum_{\ell \leq L} \frac{\lambda_f(\ell)\chi_d(\ell)x_\ell}{\sqrt{\ell}},$$

where L is a small power of K and real numbers $x_\ell \ll \log L$ are chosen so that the mollifier mimics the behaviour of $L(1/2, f \otimes \chi_d)^{-1}$. Essentially the optimal choice for the coefficients x_ℓ translates to

$$\mathcal{M}_{f,d} \sim \sum_{\ell \leq L} \frac{\mu(\ell)\lambda_f(\ell)\chi_d(\ell)}{\sqrt{\ell}} \left(1 - \frac{\log \ell}{\log L}\right)$$

Here the length of the mollifier L can be take to be at most, say, $|d|^{-1}K(\log K)^{-20}$.

The moment results required are stated in two lemmas below. These have been obtained by Iwaniec and Sarnak [19, Theorem 3.]. Throughout the text we set

$$\omega_f := \frac{2\pi^2}{2k-1} \cdot \frac{1}{L(1, \mathrm{sym}^2 f)}$$

and write $\mathcal{B}_k$ for the Hecke eigenbasis of $H_{2k}(1)$, the space of holomorphic cusp forms of weight $2k$ and full level.

Lemma 2.2. *Let h be a smooth compactly supported function on $\mathbb{R}_+$. Then there exists a small absolute constant $\delta > 0$ so that uniformly in odd fundamental discriminants d with $|d| \leq K^\delta$ we have*

$$\sum_{k \in \mathbb{Z}} h\left(\frac{k}{K}\right) \sum_{f \in \mathcal{B}_k} \omega_f L\left(\frac{1}{2}, f \otimes \chi_d\right) \mathcal{M}_{f,d} \sim K \int_0^\infty h(t)\,\mathrm{d}t.$$

Lemma 2.3. *Let h be a smooth compactly supported function on $\mathbb{R}_+$. Then there exists a small absolute constant $\delta > 0$ so that uniformly in odd fundamental discriminants d with $|d| \leq K^\delta$ we have*

$$\sum_{k \in \mathbb{Z}} h\left(\frac{k}{K}\right) \sum_{f \in \mathcal{B}_k} \omega_f L\left(\frac{1}{2}, f \otimes \chi_d\right)^2 \mathcal{M}_{f,d}^2 \sim 2K \left(\int_0^\infty h(t)\,\mathrm{d}t\right) \left(1 + \frac{\log |d|K}{\log L}\right).$$

Given these, by the Cauchy-Schwarz inequality and removing the harmonic weights by incorporating them into the mollifier we obtain that for a given fundamental discriminant d, the proportion of forms $f \in \bigcup_{k \sim K} H_{2k}(1)$ for which $L(1/2, f \otimes \chi_d) \neq 0$ is

$$\geq \frac{1}{4}\left(1 - \frac{\log |d|}{\log K}\right).$$

But as half of the central L-values vanish automatically (precisely when $(-1)^k d < 0$ due to the root number being negative) and when this does not happen the central value is proportional to the quantity $|c_g(|d|)|^2$, Proposition 2.1 follows.

From these we may conclude as follows. In our argument the precise range of uniformity in d does not matter as we need these asymptotics for fixed d. In order to obtain real zeroes the idea is to study sign changes of $c_g(|d|m^2)$ with d fixed and m traversing over the natural numbers. By the relation (2.2) we are essentially reduced to study the sign changes of the sequence

$$\lambda_f^*(p) := \lambda_f(p) - \frac{\chi_d(p)}{\sqrt{p}}$$

for p varying over primes $\ll \sqrt{K/Y}$. We will show that for a positive proportion of the forms $g \in B_{k+\frac{1}{2}}^+$ one gets a positive proportion of sign changes for the sequence $\lambda_f^*(p)$ as p traverses over the primes $p \ll \sqrt{K/Y}$. This can be done by implementing arguments of Matomäki and Radziwiłł [38] with an input on the size of the first sign change of the sequence $\lambda_f^*(p)$. Combining this with Proposition 2.1 leads to the promised amount of sign changes for a proportion $1/2-\varepsilon$ of forms, as claimed.

Remark 2.4. As indicated above, for a fixed d we are able to study sign changes along the individual sequences $c_g(|d|p^2)$ as p varies. Ideally one would want to study the sign changes of the Fourier coefficients by varying both p and d. Note that for different fundamental discriminants d the sequences $\{|d|p^2\}_p$ are disjoint, but unfortunately it seems very hard to understand how these sequences are entangled.

Concerning the first main theorem it turns out that there is another way to produce sign changes along the fundamental discriminants. Essentially, to find zeroes on individual lines δ_j it suffices to detect sign changes of $c_g(m)$ along odd integers. For this we rely on the Shimura correspondence which attaches to $g\in S^+_{k+\frac{1}{2}}(4)$ a classical integral weight Hecke cusp form f of weight $2k$ and full level. Throughout the article we normalise the Shimura lift so that its first Fourier coefficient equals one.

For detecting sign changes we use an approach, which builds upon [37, 33]. We aim to obtain sign changes of $c_g(|d|)$ along[8] squarefree $d\equiv 1 \pmod{16}$, $(-1)^k d>0$, with $1\ll |d|\ll k/Y$. The idea is to divide the long interval $]c_1, c_2k/Y[$ into short intervals $[x,x+H]$ with H as small as possible in terms of the weight and show that for many half-integral weight forms g it holds that for many $x\ll k/Y$ such a short interval $[x,x+H]$ contains a sign change of $c_g(|d|)$. For $g\in S^+_{k+\frac{1}{2}}(4)$ let us define

$$\alpha_g := \frac{\Gamma\left(k-\frac{1}{2}\right)}{(4\pi)^{k-\frac{1}{2}}\|g\|_2^2}. \tag{2.3}$$

In this normalisation the Ramanujan-Petersson conjecture for the Fourier coefficients predicts that $\sqrt{\alpha_g}c_g(m)\ll_\varepsilon k^{-1/2}(km)^\varepsilon$ with the implied constant depending only on $\varepsilon>0$. We also note that with this normalisation Waldspurger's formula (see (5.3) below) takes the form

$$\alpha_g|c_g(|d|)|^2 = \frac{L\left(\frac{1}{2}, f\otimes\chi_d\right)}{L(1,\mathrm{sym}^2 f)}\cdot\frac{2\pi^2}{2k-1}, \tag{2.4}$$

where the Shimura lift $f\in H_{2k}(1)$ is normalised so that $\lambda_f(1)=1$. It is useful to note that we have from (2.4) that

$$\alpha_g|c_g(|d|)|^2 = \omega_f L\left(\frac{1}{2}, f\otimes\chi_d\right) \tag{2.5}$$

for any fundamental discriminant d with $(-1)^k>0$. We also have

$$\sum_{f\in\mathcal{B}_k}\omega_f\sim 1. \tag{2.6}$$

Throughout the text the letter g is reserved for half-integral weight cusp forms and its Shimura lift is always denoted by the letter f.

To detect a sign change of $c_g(|d|)$ along such d with $(-1)^k d\in[x,x+H]$ it suffices to have

$$\left|\sideset{}{^\flat}\sum_{x\le(-1)^k d\le x+H} c_g(|d|)\right| < \sideset{}{^\flat}\sum_{x\le(-1)^k d\le x+H} |c_g(|d|)|,$$

where $\sum^\flat$ means summing over squarefree integers $d\equiv 1\pmod{16}$.

Because of (2.4) it is convenient to normalise by the factor $\sqrt{\alpha_g}$ and seek for the inequality

$$\left|\sideset{}{^\flat}\sum_{x\le(-1)^k d\le x+H} \sqrt{\alpha_g}c_g(|d|)\right| < \sideset{}{^\flat}\sum_{x\le(-1)^k d\le x+H} \left|\sqrt{\alpha_g}c_g(|d|)\right|,$$

[8]Here the restriction to fundamental discriminants $d\equiv 1 \pmod{16}$ is for technical convenience as it simplifies certain computations.

which naturally leads to a sign change of $c_g(|d|)$ on the interval $[x, x+H]$ as $\sqrt{\alpha_g}$ is a non-negative real number.

The following two estimates[9] suffice for the proof of the first theorem. Let us temporarily set $X = K/Y$.

Proposition 2.5. *We have*

$$\#\left\{g \in \mathcal{S}_K : \#\left\{x \sim X : \left|\sideset{}{^\flat}\sum_{x \le (-1)^k d \le x+H} \sqrt{\alpha_g} c_g(|d|)\right| \ge \sqrt{H} k^{-1/2} (\log K)^3\right\} \gg \frac{X}{(\log X)^3}\right\} \ll \frac{K^2}{(\log K)^3}.$$

Proposition 2.6. *We have*

$$\#\left\{g \in \mathcal{S}_K : \#\left\{x \sim X : \sideset{}{^\flat}\sum_{x \le (-1)^k d \le x+H} \left|\sqrt{\alpha_g} c_g(|d|)\right| \ge \frac{H}{k^{1/2}\log X}\right\} \gg \frac{X}{(\log X)^{5/2}}\right\} \gg \frac{K^2}{(\log X)^{3/2}}.$$

Indeed, it is easy to see that the first statement implies that apart from $O(K^2/(\log K)^3)$ of the forms $g \in \mathcal{S}_K$ one has the property that the inequality

$$\left|\sideset{}{^\flat}\sum_{x \le (-1)^k d \le x+H} \sqrt{\alpha_g} c_g(|d|)\right| < \sqrt{H} k^{-1/2} (\log K)^3$$

holds for almost all $x \sim X$ with the exceptional set having measure $\ll X/(\log X)^3$.

Likewise, the latter statement says that for $\gg K^2/(\log K)^{3/2}$ forms $g \in \mathcal{S}_K$ we have that

$$\sideset{}{^\flat}\sum_{x \le (-1)^k d \le x+H} \left|\sqrt{\alpha_g} c_g(|d|)\right| \ge \frac{H}{k^{1/2}\log X}$$

for $x \gg X/(\log X)^{5/2}$ of $x \sim X$. Choosing, say, $H = (\log X)^9$ and combining the previous two observations, it follows that for $\gg K^2/(\log K)^{3/2}$ of the forms $g \in \mathcal{S}_K$ one has that for $\gg X/(\log X)^{5/2}$ of the numbers $x \sim X$ the intervals $[x, x+H]$ contain a sign change of the sequence $c_g(|d|)$, leading to the promised number of real zeroes.

Both of these propositions rely crucially on the asymptotic evaluation of certain moments of quadratic twists of modular L-functions. The relevant results are the content of the following two lemmas.

Lemma 2.7. *Let ϕ and h be smooth compactly supported functions on $\mathbb{R}_+$. Then*

$$\sum_{k\in\mathbb{Z}} h\left(\frac{2k-1}{K}\right) \sum_{g \in B^+_{k+\frac12}} \alpha_g \sideset{}{^\flat}\sum_{d} |c_g(|d|)|^2 \phi\left(\frac{(-1)^k d}{X}\right) = \frac{XK}{2\pi^2}\widehat{h}(0)\widehat{\phi}(0) + O\left(K^{1+\varepsilon}X^{1/2}\right).$$

Lemma 2.8. *Let ϕ and h be smooth compactly supported functions on $\mathbb{R}_+$. Then*

$$\begin{aligned}&\sum_{k\in\mathbb{Z}} h\left(\frac{2k-1}{K}\right) \sum_{g \in B^+_{k+\frac12}} \alpha_g^2 \omega_g^{-1} \sideset{}{^\flat}\sum_{d} |c_g(|d|)|^4 \phi\left(\frac{(-1)^k d}{X}\right)\\ &= \frac{XK}{\pi^2}\Bigg(\mathcal{G}(0)\widehat{h}(0)\widehat{\phi}(0)\log(XK)\\ &\qquad + \mathcal{G}(0)\widehat{h}(0)\int_{\mathbb{R}} \phi(\xi)\log\xi\,\mathrm{d}\xi + \mathcal{G}(0)\widehat{\phi}(0)\int_{\mathbb{R}} h(\xi)\log\xi\,\mathrm{d}\xi + \widehat{\phi}(0)\widehat{h}(0)\cdot\mathcal{C}\Bigg) + O(K^{1+\varepsilon}X^{3/4}),\end{aligned}$$

where

$$\mathcal{G}(0) = \prod_{p\neq 2}\left(1 - \frac{1}{p^2}\left(\frac{1-p^{-1}}{1-p^{-2}}\right)\right)$$

and the constant $\mathcal{C}$ is given in (9.6).

[9]Sometimes we write $\#\mathcal{X}$ for the cardinality of a set $\mathcal{X}$.

In the latter lemma and throughout the paper we write as an abuse of notation ω_g for ω_f when f is the Shimura lift of g.

Proofs of these results are reasonably standard using the state-of-the-art methods, but on the other hand they are also rather involved. It is worth to mention that in the proof of Lemma 2.8 a nice structural feature that the off-diagonal contribution arising from an application of Petersson's formula cancels part of the diagonal contribution from the application of the same formula. This gives a new instance of a similar phenomenon present in some prior works [4, 5, 22].

A few remarks are in order concerning the various averages in the preceding two lemmas and in particular to explain why they are required. Ideally we would like to evaluate the moments

$$\sum_{(-1)^k d\sim X}^{\flat} |c_g(|d|)|^j$$

for $j \in \{2,4\}$ uniformly in terms of the weight $k+\frac{1}{2}$ of g. By Waldspurger's formula these reduce to asymptotic evaluation of averaged first and second moments of $L(1/2, f\otimes\chi_d)$ with uniformity in k. More precisely, we are required to understand the asymptotic behaviour of the sums

$$\sum_{(-1)^k d\sim X}^{\flat} L\left(\frac{1}{2}, f\otimes\chi_d\right) \qquad \text{and} \qquad \sum_{(-1)^k d\sim X}^{\flat} L\left(\frac{1}{2}, f\otimes\chi_d\right)^2, \tag{2.7}$$

where $f\in H_{2k}(1)$ is the Shimura lift of g.

Recall that the complexity of a moment problem is measured by the ratio between logarithm of the analytic conductor and logarithm of the family size. Denote this ratio by r. The situations where $r=4$ is the edge of current technology where one can hope to obtain an asymptotic formula with a power saving error term. However, usually in this case we often barely fail to produce such asymptotics and a deep input is typically required in the rare cases when an asymptotic formula can be obtained. Note that in the latter sum in (2.7) the ratio between the logarithm of the conductor and the logarithm of the family size is greater than four. Moreover, in our applications the parameter X will be small, $X \ll \sqrt{K}$. In this situation the summation range is too short to evaluate even the first moment.

However, one can remedy the situation by introducing additional averages. Averaging over $g\in B^+_{k+\frac{1}{2}}$ brings us to the situation where the ratio r is precisely 4 in the second moment problem and in this case it is plausible that the methods from Li's recent breakthrough [34] (which builds upon [46]) can be adapted to our setting to yield the expected asymptotics, but saving only a power of a logarithm in the error term. Moreover, a direct adaptation of this method cannot handle the introduction of a mollifier of length $\gg |d|^\varepsilon$. For all these reasons we have added one more averaging over the weights, which which brings us to the situation where $2<r<3$.

Computations of these moments are somewhat related to the work of Khan [22] concerning the central L-values of symmetric square lift and we are able to use some of his computations. We remark that the asymptotic formula required in Lemma 2.7 can be established without the additional k-average.

2.1. **Possible extensions.** It is natural to wonder whether we can improve the result of Theorem 1.1 to hold for a positive proportion of forms using a mollifier. By Waldspurger's formula a natural choice for the mollifier to study sign changes of $c_g(|d|)$ would be a quantity $\mathcal{M}_g(d)$, which is a truncated Dirichlet series approximation for $L(1/2, f\otimes\chi_d)^{-1/2}$. A natural approach would be to start by letting the mollifying factor for $c_g(|d|)$ to be of the form

$$\left(\sum_{\ell\le L}\frac{x_\ell\mu(\ell)\lambda_f(\ell)\chi_d(\ell)}{\sqrt{\ell}}\right)^2$$

for some real coefficients x_ℓ to guarantee its non-negativity which is crucial for detecting sign changes. For this mollifier we would need to choose the coefficients x_ℓ so that

$$\sum_{\ell\le L}\frac{x_\ell\mu(\ell)\lambda_f(\ell)\chi_d(\ell)}{\sqrt{\ell}} \approx L\left(\frac{1}{2}, f\otimes\chi_d\right)^{-1/4},$$

but unfortunately it seems unclear how to do this effectively. Moreover, a crucial step in the proof of Proposition 2.5 would involve understanding sums of the form

$$\sum_{g\in B^+_{k+\frac{1}{2}}} \alpha_g c_g(|d|)c_g(|d|+h)\mathcal{M}_g(d)\mathcal{M}_g(d+h). \tag{2.8}$$

In this case, after several applications of Hecke relations, studying (2.8) reduces to understanding the sums

$$\sum_{g\in B^+_{k+\frac{1}{2}}} \alpha_g c_g(|d|)c_g(|d|+h)\lambda_f(\ell).$$

This sum can be made amenable for Lemma 5.4 by inverting the relation (5.1) to write $c_g(|d|)\lambda_f(\ell)$ as a linear combination of Fourier coefficients of g at various arguments (at least for squarefree ℓ). Thus it should be possible to prove a variant of Proposition 2.5 with this mollifier (with general coefficients x_ℓ). On the other hand, now evaluating the mollified moments

$$\sum_{k\in\mathbb{Z}} h\left(\frac{2k-1}{K}\right) \sum_{g\in B^+_{k+\frac{1}{2}}} \alpha_g \sideset{}{^\flat}\sum_{d} |c_g(|d|)|^2\mathcal{M}_g(d)^2\phi\left(\frac{(-1)^k d}{X}\right),$$

$$\sum_{k\in\mathbb{Z}} h\left(\frac{2k-1}{K}\right) \sum_{g\in B^+_{k+\frac{1}{2}}} \alpha_g^2\omega_g^{-1} \sideset{}{^\flat}\sum_{d} |c_g(|d|)|^4\mathcal{M}_g(d)^4\phi\left(\frac{(-1)^k d}{X}\right)$$

and optimising them becomes technically much more challenging compared to the corresponding moments with the Iwaniec-Sarnak mollifier $\mathcal{M}_{f,d}$ [19].

Related to this, again by a repeated application of Hecke relations, a key step towards evaluating these mollified moments is to evaluating the twisted analogues of the sums appearing in Lemmas 2.7 and 2.8. Indeed, with some additional work one may show by our methods that there exists an absolute constant $\delta > 0$ so that uniformity in $\ell \le K^\delta$ we can obtain asymptotics for

$$\sum_{k\in\mathbb{Z}} h\left(\frac{2k-1}{K}\right) \sum_{g\in B^+_{k+\frac{1}{2}}} \alpha_g \sideset{}{^\flat}\sum_{d} |c_g(|d|)|^2\lambda_f(\ell)\chi_d(\ell)\phi\left(\frac{(-1)^k d}{X}\right)$$

and

$$\sum_{k\in\mathbb{Z}} h\left(\frac{2k-1}{K}\right) \sum_{g\in B^+_{k+\frac{1}{2}}} \alpha_g^2\omega_g^{-1} \sideset{}{^\flat}\sum_{d} |c_g(|d|)|^4\lambda_f(\ell)\chi_d(\ell)\phi\left(\frac{(-1)^k d}{X}\right).$$

However, it is hard to benefit from this due to lack of understanding on how choose the coefficients x_ℓ optimally in the mollifier.

It might be possible to make this kind of reasoning work, but in order to keep the article reasonably short and avoiding various arduous technical/notational complications, we have decided not to pursue this approach in the present work. The approach outlined introduces a library of additional variable and intricate cross-conditions involving various variables that seem troublesome to deal with. We plan to investigate this possibility in a future work.

It is also interesting to work out whether the methods of the present paper can be combined with mollifying using an Euler product [47, 14, 41] as in the the work of Lester and Radziwiłł to prove a result that holds for a positive proportion of forms, of course assuming GRH. We emphasise that the purpose of the present article is to demonstrate that one is able to deduce highly non-trivial information about the zeroes of half-integral weight Hecke cusp forms using methods very different from those used in the integral weight case that do not produce strong results in the present setting.

2.2. **Organisation of the article.** This paper is organised as follows. In Section 5. we gather basic facts about half-integral weight modular forms and other auxiliary results we need. In Section 6. we show how to reduce the question of finding real zeroes to studying the sign changes of the Fourier coefficients. The next sections are devoted to the proofs of Lemmas 2.7 and 2.8. These are then used to prove Propositions 2.5 and 2.6 from which Theorem 1.1 is deduced. In Section 11. the proof Theorem 1.3 is completed by studying sign changes using the ideas from multiplicative number theory.

3. Acknowledgements

This work was supported by the Emil Aaltonen Foundation, the Finnish Cultural Foundation, and the Engineering and Physical Sciences Research Council [grant number EP/T028343/1]. The author would like to thank Steve Lester and Kaisa Matomäki for useful discussions. A part of this paper was written while the author was visiting the Mittag-Leffler Institute in Djursholm, Sweden, whose excellent working conditions are acknowledged.

4. Notation

We use standard asymptotic notation. If f and g are complex-valued functions defined on some set, say $\mathcal{D}$, then we write $f \ll g$ to signify that $|f(x)| \leqslant C|g(x)|$ for all $x \in \mathcal{D}$ for some implicit constant $C \in \mathbb{R}_+$. The notation $O(g)$ denotes a quantity that is $\ll g$, and $f \asymp g$ means that both $f \ll g$ and $g \ll f$. We write $f = o(g)$ if g never vanishes in $\mathcal{D}$ and $f(x)/g(x) \longrightarrow 0$ as $x \longrightarrow \infty$. The notation $f = \Omega(g)$ means that $f \neq o(g)$. The letter ε denotes a positive real number, whose value can be fixed to be arbitrarily small, and whose value can be different in different instances in a proof. All implicit constants are allowed to depend on ε, on the implicit constants appearing in the assumptions of theorem statements, and on anything that has been fixed. When necessary, we will use subscripts $\ll_{\alpha,\beta,\ldots}, O_{\alpha,\beta,\ldots}$, etc. to indicate when implicit constants are allowed to depend on quantities $\alpha, \beta, \ldots$

We define $\chi_d(\cdot) := \left(\frac{d}{\cdot}\right)$, the Kronecker symbol, for all non-zero integers d. Let us also write $1_{m=n}$ for the characteristic function of the event $m = n$. Furthermore, $\mathrm{Re}(s)$ and $\mathrm{Im}(s)$ are the real- and imaginary parts of $s \in \mathbb{C}$, respectively, and occasionally we write σ for $\mathrm{Re}(s)$. We write $e(x) := e^{2\pi i x}$. For ϕ a compactly supported smooth function, we define the Fourier transform $\widehat{\phi}(y)$ of ϕ by

$$\widehat{\phi}(y) := \int_{\mathbb{R}} \phi(x) e(-xy) \, \mathrm{d}x.$$

We write φ for the Euler totient function. If a and b are integers we write $[a, b]$ for their least common multiple and (a, b) for their greatest common divisor. It will always be clear from context whether $[a, b]$, say, denotes a least common multiple or a real interval. The sum $\sum_{a\,(c)}^{*}$ means that the summation is over residue classes coprime to the modulus. Given coprime integers a and c, we write $\overline{a} \,(\mathrm{mod}\ c)$ for the multiplicative inverse of a modulo c. As usual, $\zeta(s)$ denotes the Riemann zeta function. Finally, $\sum^{\flat}$ means we are summing over all fundamental discriminants $\equiv 1 \,(\mathrm{mod}\ 16)$.

5. Preliminaries

5.1. **Half-integral weight forms.** The group $\mathrm{SL}_2(\mathbb{R})$ acts on the upper half-plane $\mathbb{H}$ by $\gamma.z := \frac{az+b}{cz+d}$, where $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $z = x + iy \in \mathbb{C}$. Let $\Gamma_0(4)$ denote the congruence subgroup consisting of matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ in $\mathrm{SL}_2(\mathbb{Z})$ such that c is divisible by 4.

Let $\theta(z) := \sum_{n=-\infty}^{\infty} e(n^2 z)$ denote the standard theta function on $\mathbb{H}$. If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(4)$, we have $\theta(Az) = j(A, z)\theta(z)$, where $j(A, z)$ is the so-called theta multiplier. For an explicit formula for $j(A, z)$, see [45, 1.10]. Fix a positive integer k. Let $S_{k+\frac{1}{2}}(4)$ denote the space of holomorphic cusp forms of weight $k + \frac{1}{2}$ for the group $\Gamma_0(4)$. This means that a function $g : \mathbb{H} \longrightarrow \mathbb{C}$ belongs to $S_{k+\frac{1}{2}}(4)$ if

- $g(Az) = j(A, z)^{2k-1} g(z)$ for every $A \in \Gamma_0(4)$.

- g is holomorphic.
- g vanishes at the cusps.

Any such form g has a Fourier expansion of the form

$$g(z) = \sum_{n=1}^{\infty} c_g(n) n^{\frac{k}{2}-\frac{1}{4}} e(nz),$$

where $c_g(n)$ are the Fourier coefficients of g.

For $g, h \in S_{k+\frac{1}{2}}(4)$, we define the Petersson inner product $\langle g, h\rangle$ to be

$$\langle g, h\rangle := \frac{1}{[\mathrm{SL}_2(\mathbb{Z}) : \Gamma_0(4)]} \int\limits_{\Gamma_0(4)\backslash\mathbb{H}} g(z)\overline{h}(z) y^{k+\frac{1}{2}} \frac{\mathrm{d}x\,\mathrm{d}y}{y^2}.$$

For any odd prime p there exists a Hecke operator $T(p^2)$ acting on the space of half-integral weight modular forms given by

$$T(p^2)g(z) := \sum_{n=1}^{\infty} \left(c_g(p^2 n) + \left(\frac{n}{p}\right) p^{k-\frac{3}{2}} c_g(n) + p^{2k} c_g\left(\frac{n}{p^2}\right)\right) e(nz).$$

Here we have used the convention that $b(x) = 0$ unless $x \in \mathbb{Z}$. We call a half-integral weight cusp form a Hecke cusp form if $T(p^2)g = \gamma_g(p) g$ for all $p > 2$ for some $\gamma_g(p) \in \mathbb{C}$.

The Kohnen plus subspace $S^{+}_{k+\frac{1}{2}}(4) \subset S_{k+\frac{1}{2}}(4)$ consists of all weight $k+\frac{1}{2}$ Hecke cusp forms whose n^{th} Fourier coefficient vanishes whenever $(-1)^k n \equiv 2, 3 \pmod 4$. This space has a basis consisting of simultaneous eigenfunctions of the $T(p^2)$ for odd p. As $k \longrightarrow \infty$, asymptotically one third of half integral weight cusp forms lie in the Kohnen plus space by dimension considerations. In this space Shimura's correspondence [45] between half-integral weight forms and integral weight forms is well-understood.

Indeed, Kohnen proved [27] that there exists a Hecke algebra isomorphism between $S^{+}_{k+\frac{1}{2}}(4)$ and the space of level 1 cusp forms of weight $2k$. That is, $S^{+}_{k+\frac{1}{2}}(4) \simeq H_{2k}(1)$ as Hecke modules. Also recall that every Hecke cusp form $g \in S^{+}_{k+\frac{1}{2}}(4)$ can be normalised so that it has real Fourier coefficients and throughout the article we assume that g has been normalised in this way.

For a fundamental discriminant d with $(-1)^k d > 0$ we know that

$$c_g(n^2|d|) = c(|d|) \sum_{r|n} \frac{\mu(r)\chi_d(r)}{\sqrt{r}} \lambda_f\left(\frac{n}{r}\right), \tag{5.1}$$

where $\lambda_f(n)$ denotes the n^{th} Hecke eigenvalue of the Shimura lift f (see equation (2) of [26]). In particular, if p is a prime this becomes

$$c_g(|d|p^2) = c(|d|) \left(\lambda_f(p) - \frac{\chi_d(p)}{\sqrt{p}}\right). \tag{5.2}$$

The proof of our first main result uses the explicit form of Waldspurger's formula due to Kohnen and Zagier [26].

Lemma 5.1. *For a Hecke cusp form $g \in S^{+}_{k+\frac{1}{2}}(4)$ we have*

$$|c_g(|d|)|^2 = L\left(\frac{1}{2}, f \otimes \chi_d\right) \cdot \frac{(k-1)!}{\pi^k} \cdot \frac{\langle g, g\rangle}{\langle f, f\rangle}. \tag{5.3}$$

for each fundamental discriminant d with $(-1)^k d > 0$, where f is a holomorphic modular form attached to g via Shimura correspondence, normalised so that $\lambda_f(1) = 1$.

We also remark that $L(1/2, f \otimes \chi_d)$ vanishes when $(-1)^k d < 0$ due to the sign in the functional equation. It follows directly from (5.3) that $L(1/2, f \otimes \chi_d) \geq 0$ otherwise.

The approximate functional equation is also needed. This follows by an easy modification of [41, Lemma 5].

Lemma 5.2. *Let f be a Hecke cusp form of weight $2k$ for the full modular group $\mathrm{SL}_2(\mathbb{Z})$. Then*

$$L\left(\frac{1}{2}, f\otimes\chi_d\right) = 2\sum_{m=1}^{\infty}\frac{\lambda_f(m)\chi_d(m)}{\sqrt{m}}V_k\left(\frac{m}{|d|}\right),$$

where

$$V_k(x) := \frac{1}{2\pi i}\int_{(\sigma)} g(s)x^{-s}e^{s^2}\frac{\mathrm{d}s}{s} \quad \text{with} \quad g(s) := (2\pi)^{-s}\frac{\Gamma(s+k)}{\Gamma(k)}. \tag{5.4}$$

Furthermore, for $|\mathrm{Im}(s)|\le\sqrt{k}$ we have for any $A\ge 1$ that

$$V_k(\xi) = 1 + O(\xi^{1/2-o(1)}), \qquad V_k(\xi) = 1 + O_A(\xi^{-A}). \tag{5.5}$$

We also have the estimates

$$V_k(\xi) \ll_A \left(\frac{k}{\xi}\right)^A,$$

$$V_k^{(B)}(\xi) \ll_{A,B} \xi^{-B}\left(\frac{k}{\xi}\right)^A$$

for any $A>0$ and integer $B\ge 0$.

In addition, using Stirling's formula there exists a holomorphic function $R(s,k)$ so that for $\mathrm{Re}(s)\ge -k/2$ *we have $R(s,k)\ll |s|^2/k$ and*

$$\frac{\Gamma(s+k)}{\Gamma(k)} = k^s e^{R(s,k)}\left(1+O\left(k^{-1}\right)\right). \tag{5.6}$$

In particular, for $|\mathrm{Re}(s)|\le\sqrt{k}$ we have

$$\frac{\Gamma(s+k)}{\Gamma(k)} = k^s\left(1+O\left(\frac{|s|^2}{k}\right)\right).$$

From this we have the useful approximation

$$V_k(x) = \frac{1}{2\pi i}\int_{(A)}(2\pi)^{-s}\left(\frac{k}{x}\right)^s\frac{\mathrm{d}s}{s} + O_\varepsilon\left(x^{-\varepsilon}k^{-1+\varepsilon}\right) \tag{5.7}$$

for any $A>0$.

Note also that $V_k = V_{Ku+1}$ for $u=(k-1)/K$.

5.2. **Basic tools.**

5.2.1. *Summation formulas.* One of the most important tools is the Petersson trace formula.

Lemma 5.3. *Let m and n be natural numbers, and k be a positive integer. Then*

$$\sum_{f\in\mathcal{B}_k}\omega_f\lambda_f(m)\lambda_f(n) = 1_{m=n} + 2\pi i^{2k}\sum_{c=1}^{\infty}\frac{S(m,n;c)}{c}J_{2k-1}\left(\frac{4\pi\sqrt{mn}}{c}\right),$$

where $S(m,n;c)$ is the usual Kloosterman sum and J_ν is the J-Bessel function.

A half-integral weight analogue for this is also needed. The following result for forms in the Kohnen plus subspace is [6, Lemma 6.]. Recall the definition of the normalising factor α_g from (2.3).

Lemma 5.4. *Let $k\ge 3$ be an integer. Let m,n be positive integers with $(-1)^k m$, $(-1)^k n\equiv 0, 1\,(\mathrm{mod}\ 4)$ and set $\kappa := k+\frac{1}{2}$. Then*

$$\sum_{g\in B^+_{k+\frac{1}{2}}}\alpha_g c_g(m)c_g(n) = \frac{2}{3}\left(1_{m=n} + 2\pi e\left(-\frac{\kappa}{4}\right)\sum_c\frac{K^+_\kappa(m,n;c)}{c}J_{k-\frac{1}{2}}\left(\frac{4\pi\sqrt{mn}}{c}\right)\right),$$

where for $m,n\in\mathbb{Z}$ and $c\in\mathbb{N}$, we define the modified Kloosterman sum as

$$K^{+}_{\kappa}(m,n;c) := \sum_{d\,(c)}^{*}\epsilon_d^{2\kappa}\left(\frac{c}{d}\right)e\left(\frac{md+n\overline{d}}{c}\right)\cdot\begin{cases}0 & \text{if } 4\nmid c\\ 2 & \text{if } 4|c,\, 8\nmid c\\ 1 & \text{if } 8|c\end{cases}$$

Here

$$\epsilon_d := \begin{cases}1 & \text{if } d\equiv 1\,(4)\\ i & \text{if } d\equiv 3\,(4)\end{cases}$$

Note that the sum K^{+}_{κ} is 2-periodic in κ and it satisfies a Weil-type bound (see [48] and [18, Section 1.6.])

$$|K^{+}_{\kappa}(m,n;c)| \le d(c)(m,n,c)^{1/2}c^{1/2}, \tag{5.8}$$

where $d(c)$ is the ordinary divisor function.

Let us now define several integral transforms. For a smooth compactly supported function h, we set

$$\hbar(y) := \int_0^\infty \frac{h(\sqrt{u})}{\sqrt{2\pi u}}e^{iyu}\,\mathrm{d}u.$$

Define

$$W^{(1)}_K(m,v) := \int_0^\infty \frac{V_{\sqrt{u}K+1}(m)h(\sqrt{u})}{\sqrt{2\pi u}}e^{iuv}\,\mathrm{d}u,$$

and

$$W^{(2)}_K(m,n,v) := \int_0^\infty \frac{V_{\sqrt{u}K+1}(m)V_{\sqrt{u}K+1}(n)h(\sqrt{u})}{\sqrt{2\pi u}}e^{iuv}\,\mathrm{d}u.$$

Main properties of these integral transforms have been worked out by Khan. He has shown [22, (2.17)] that

$$W^{(2)}_K(n,m,v) \ll_{A_1,A_2,B} \left(\frac{K}{n}\right)^{A_1}\left(\frac{K}{m}\right)^{A_2}v^{-B}$$

for any $A_1,A_2>0$ and $B\ge 0$. Thus $W^{(2)}_K(n,m,v)$ is essentially supported on $n\le K^{1+\varepsilon}$, $m\le K^{1+\varepsilon}$ and $v\le K^{\varepsilon}$. Moreover, we have estimates for the derivatives;

$$\frac{\partial^j}{\partial\xi^j}W^{(2)}_K\left(\xi,b,\frac{c}{\xi}\right) \ll_{j,A,B,C} \xi^{-j-A}b\left(\frac{\xi}{c}\right)^{B}b^{-C}K^{\varepsilon} \tag{5.9}$$

for any $j\ge 0$, $A,B,C>0$.

Similarly [22, (2.25)] we have

$$W^{(1)}_K(n,v) \ll_{A,B} \left(\frac{K}{n}\right)^{A}v^{-B} \tag{5.10}$$

for any $A>0$ and $B\ge 0$.

We also have the identities [22, (2.14), (2.24)]

$$2\sum_{k\equiv 0\,(2)} i^k h\left(\frac{k-1}{K}\right)V_k(n)J_{k-1}(t) = -\frac{K}{\sqrt{t}}\mathrm{Im}\left(e^{-2\pi i/8}e^{it}W^{(1)}_K\left(n,\frac{K^2}{2t}\right)\right) + O_A\left(\frac{t}{K^4}\left(\frac{K}{n}\right)^{A}\right) \tag{5.11}$$

and

$$2\sum_{k\equiv 0\,(2)} i^k h\left(\frac{k-1}{K}\right) V_k(n)V_k(m)J_{k-1}(t)$$

(5.12)

$$= -\frac{K}{\sqrt{t}}\mathrm{Im}\left(e^{-2\pi i/8}e^{it}W_K^{(2)}\left(n,m,\frac{K^2}{2t}\right)\right) + O\left(\frac{t}{K^4}\int_{\mathbb{R}} v^4\left|\int_0^\infty V_{uK+1}(n)V_{uK+1}(m)h(u)e^{iuv}\,\mathrm{d}u\right|\,\mathrm{d}v\right).$$

A simple application of the Poisson summation gives

$$\sum_{k\equiv 0\,(2)} h\left(\frac{k-1}{K}\right) = \frac{K}{2}\widehat{h}(0) + O_B(K^{-B})$$

for every $B > 0$.

For real $\xi_1 > 0$ and $\xi_2 > 0$ we define

$$W(\xi_1,\xi_2,v) := \frac{1}{(2\pi i)^2}\int_{(A_1)}\int_{(A_2)} (2\pi)^{-x-y}e^{x^2+y^2}\xi_1^{-x}\xi_2^{-y}\hbar_{x+y}(v)\,\frac{\mathrm{d}x\,\mathrm{d}y}{xy},$$

where

$$\hbar_z(v) := \int_0^\infty \frac{h(\sqrt{u})}{\sqrt{2\pi u}}u^{z/2}e^{iuv}\,\mathrm{d}u.$$

Then

$$W_K^{(2)}(m,n,v) = W\left(\frac{m}{K},\frac{n}{K},v\right) + O\left(K^{-1+\varepsilon}\right). \tag{5.13}$$

Integrating by parts shows that $\hbar_z(v) \ll_{\mathrm{Re}(z),B} (1+|z|)^B v^{-B}$ and consequently

$$W(\xi_1,\xi_2,v) \ll_{B,A_1,A_2} \xi_1^{-A_1}\xi_2^{-A_2}v^{-B}$$

for $A_1, A_2 > 0$ and $B \geq 0$.

The treatment of the off-diagonals in the moment computations requires the following auxiliary result [22, Lemma 3.3.] concerning the properties of the Mellin transform of the function $\hbar_z$.

Lemma 5.5. *For $0 < \mathrm{Re}(s) < 1$, we have*

$$\widetilde{\hbar}_z(s) = \int_0^\infty \frac{h(\sqrt{u})}{\sqrt{2\pi u}}u^{z/2-s}\Gamma(s)\left(\cos\left(\frac{\pi s}{2}\right) + i\sin\left(\frac{\pi s}{2}\right)\right)\,\mathrm{d}u \tag{5.14}$$

and the bound $\widetilde{\hbar}_z(s) \ll_{\mathrm{Re}(z)} (1+|z|)^3|s|^{-2}$.

For $0 < c < 1$ we have

$$\hbar_z(v) = \frac{1}{2\pi i}\int_{(c)} v^{-s}\widetilde{\hbar}_z(s)\,\mathrm{d}s. \tag{5.15}$$

Another key tool is the Poisson summation formula.

Lemma 5.6. *Let f be a Schwartz function and a be a residue class modulo c. Then*

$$\sum_{n\equiv a\,(mod\ c)} f(n) = \frac{1}{c}\sum_n \widehat{f}\left(\frac{n}{c}\right)e\left(\frac{an}{c}\right),$$

where $\widehat{f}$ denotes the Fourier transform of f. Note that this reduces to the classical Poisson summation formula when $c = 1$.

We shall also need a different variant of the Poisson summation formula. For this, let us define, for any $j \in \mathbb{Z}$, a Gauss-type sum

$$\tau_j(n) := \sum_{b\,(n)} \left(\frac{n}{b}\right) e\left(\frac{jb}{n}\right) = \left(\frac{(1+i)}{2} + \left(\frac{-1}{n}\right)\frac{(1-i)}{2}\right) G_j(n), \tag{5.16}$$

where

$$G_j(n) := \left(\frac{(1-i)}{2} + \left(\frac{-1}{n}\right)\frac{(1+i)}{2}\right) \sum_{a\,(n)} \left(\frac{n}{a}\right) e\left(\frac{aj}{n}\right).$$

The twisted Poisson summation formula needed for our purposes is contained in the following lemma.

Lemma 5.7. *([41, Lemma 7.]) Let n be an odd integer and q positive integer so that $(n,q)=1$. Suppose that F is a smooth and compactly supported function on $\mathbb{R}$. Finally, let η be a reduced residue class modulo q. Then*

$$\sum_{d\equiv\eta\,(q)} \left(\frac{d}{n}\right) F(d) = \frac{1}{qn}\left(\frac{q}{n}\right) \sum_{\ell\in\mathbb{Z}} \widehat{F}\left(\frac{\ell}{nq}\right) e\left(\frac{\ell\eta\overline{n}}{q}\right) \tau_\ell(n),$$

where $\widehat{F}$ is the usual Fourier transform.

The Gauss-type sum in the previous lemma may be evaluated explicitly.

Lemma 5.8. *If m and n are coprime odd integers, then $\tau_\ell(mn) = \tau_\ell(m)\tau_\ell(n)$. Furthermore, if p^α is the highest power of p that divides ℓ (setting $\alpha=\infty$ if $\ell=0$), then*

$$\tau_\ell\left(p^\beta\right) = \begin{cases} 0 & \text{if } \beta \le \alpha \text{ is odd} \\ \varphi(p^\beta) & \text{if } \beta \le \alpha \text{ is even} \\ -p^\alpha & \text{if } \beta = \alpha+1 \text{ is even} \\ \left(\frac{\ell p^{-\alpha}}{p}\right) p^\alpha\sqrt{p} & \text{if } \beta = \alpha+1 \text{ is odd} \\ 0 & \text{if } \beta \ge \alpha+2 \end{cases} \tag{5.17}$$

5.3. **Other tools.** We also record the following well-known uniform estimate for the J-Bessel function. For $\nu \ge 0$ and $x > 0$, the J_ν-Bessel function satisfies the bound

$$J_\nu(x) \ll \frac{x}{\sqrt{\nu}}\left(\frac{ex}{2\nu}\right)^\nu. \tag{5.18}$$

Another crucial auxiliarly result is the stationary phase method for estimating oscillatory exponential integrals. We quote the following result [8] of Blomer, Khan, and Young that is uniform with respect to multiple parameters.

Lemma 5.9. *Let $X, Y, V, V_1, Q > 0$ and $Z := Q + X + Y + V_1 + 1$, and assume that*

$$Y \ge Z^{3/20}, \qquad V_1 \ge V \ge \frac{QZ^{1/40}}{Y^{1/2}}.$$

Suppose that h is a smooth function on $\mathbb{R}$ with support on an interval J of length V_1, satisfying

$$h^{(j)}(t) \ll_j XV^{-j}$$

for all $j \in \mathbb{N}\cup\{0\}$. Suppose that f is a smooth function on J such that there exists a unique point $t_0 \in J$ such that $f'(t_0)=0$, and furthermore

$$f''(t) \gg YQ^{-2}, \quad f^{(j)}(t) \ll_j YQ^{-j} \quad \text{for all } j \ge 1 \text{ and } t \in J.$$

Then

$$\int_{\mathbb{R}} h(t)e(f(t))\,\mathrm{d}t = e^{sgn(f''(t_0))\pi i/4}\frac{e(f(t_0))}{\sqrt{|f''(t_0)|}} h(t_0) + O\left(\frac{Q^{3/2}X}{Y^{3/2}}\cdot\left(V^{-2} + (Y^{2/3}/Q^2)\right)\right).$$

In particular, we have the trivial bound

$$\int_{\mathbb{R}} h(t)e(f(t))\,\mathrm{d}t \ll \frac{XQ}{\sqrt{Y}} + 1.$$

Our final tool used to detect sign changes of the Fourier coefficients in the proof of Theorem 1.3 is the following result of Chen, Wu, and Zhang concerning the first sign change of the sequence $\lambda_f^*(p) = \lambda_f(p) - \chi_d(p)/\sqrt{p}$. Let p_f be the smallest prime for which $\lambda_f^*(p_f) < 0$.

Lemma 5.10 (Corollary 1 in [10])**.** *There exists an absolute constant $c > 0$ so that for all except $\ll k\exp(-c(\log k)/\log\log k)$ forms $f \in H_{2k}(1)$ we have $p_f \ll \log k$.*

6. Reduction to the study of Fourier coefficients

Our approach to relate detecting real zeroes to properties of the Fourier coefficients follows the previous works [13, 36, 31] in the setting of integral weight cusp forms. The main result in this direction is the following observation, which is a half-integral weight analogue for [13, Theorem 3.1.].

Proposition 6.1. *Let $\alpha \in \{-\frac{1}{2}, 0\}$. Then there are positive constants c_1, c_2 and η such that, for all integers $\ell \in]c_1, c_2\sqrt{k/\log k}[$ and all Hecke eigenforms $g \in S^+_{k+\frac{1}{2}}(4)$, we have*

$$\sqrt{\alpha_g}\left(\frac{e}{2\ell}\right)^{\frac{k}{2}-\frac{1}{4}} g(\alpha + iy_\ell) = \sqrt{\alpha_g}c_g(\ell)e(\alpha\ell) + O(k^{-1/2-\eta}),$$

where $y_\ell := (k-1/2)/4\pi\ell$ and the implicit constant in the error term is absolute.

The proof is the same as in [13]. The only automorphic input used in the proof, besides the existence of the Fourier expansion, is Deligne's bound for the Hecke eigenvalues, an analogue of which is not known for half-integral weight forms. However, it turns out that the pointwise bound[10] $c(n) \ll_\varepsilon n^{1/6+\varepsilon}$ of Conrey and Iwaniec [11, Corollary 3] suffices when the k-dependence is made explicit. By the preceding footnote we have

$$\sqrt{\alpha_g}|c_g(m)| \ll_\varepsilon m^\varepsilon\sqrt{\alpha_g}|c_g(\tilde{m})|,$$

where $\tilde{m}$ is the squarefree kernel of m and the implied constant depends only on $\varepsilon > 0$. To estimate the latter factor on the right-hand side one combines the works of Young [49] and Petrow-Young [40] to see that for any squarefree[11] integer q one has $L(1/2, f\otimes\chi_q) \ll_\varepsilon (kq)^{1/3+\varepsilon}$ uniformly in both k and q. Together with computations of Mao [3, Appendix 2] and a slight extension of these that appeared in [7, Section 9] this leads to the following lemma.

Lemma 6.2. *Let $g \in S^+_{k+\frac{1}{2}}(4)$ be a Hecke eigenform. Then we have*

$$\sqrt{\alpha_g}c_g(m) \ll_\varepsilon k^{-1/3+\varepsilon}m^{1/6+\varepsilon} \tag{6.1}$$

for any $\varepsilon > 0$.

Recall that here the most optimistic bound would be $\ll_\varepsilon k^{-1/2}(km)^\varepsilon$. The idea is that the function $\xi \mapsto \xi^k e^{-k}$ has a maximum at $\xi = k$ and is very localised there. We give a fairly detailed argument as the k-dependence in (6.1) is crucial and also remark that if the exponent of k in Lemma 6.2 would be large than $-1/12$, then the method used to obtain the required approximation (Proposition 6.1) would not succeed. This stems from treating the error term coming from Lemma 6.3 applied to the partial sum $\Phi_g^{(2)}$.

Let us set

$$I_s(y) := y^{(s-1)/2}e^{-y}$$

for any $y > 0$ and $s \in \mathbb{C}$. The following auxiliary result will be useful.

---

[10] Note that in general the estimate in [11] only holds for squarefree n. However, in our case g is a Hecke eigenform and this case the estimate can be extended for arbitrary $n \in \mathbb{N}$, see [21, Lemma 3.3.].

[11] Note that in [49] the conductor of the quadratic character was assumed to be odd, but the same result holds for even discriminants as well [1, Appendix A].

Lemma 6.3. *([13, Lemma 2.3.]) For $|h| \ll k^{2/3-\delta}$, with any $\delta > 0$, we have*

$$I_{k+\frac{1}{2}}\left(\frac{k}{2}-\frac{1}{4}+h\right) = I_{k+\frac{1}{2}}\left(\frac{k}{2}-\frac{1}{4}\right) e^{-h^2/(k-1/2)}\left(1+O\left(k^{-3\delta}\right)\right).$$

With this at hand we are ready to prove the approximation of the values of g at certain arguments.

Proof of Proposition 6.1. Throughout the proof, let y be a real parameter with $\sqrt{k} \ll y \ll k$. Furthermore, let $\varepsilon > 0$ be arbitrarily small but fixed. We freely refer to the argument of Ghosh and Sarnak for details. Just by the Fourier expansion (2.1) we have

$$\begin{aligned}\sqrt{\alpha_g}g(\alpha+iy) &= \sum_{m=1}^{\infty} \sqrt{\alpha_g}c_g(m)m^{\frac{k}{2}-\frac{1}{4}}e(n\alpha)e^{-2\pi my}\\ &= (2\pi y)^{-\frac{k}{2}+\frac{1}{4}}\Phi_g(k;\alpha,y),\end{aligned}$$

where

$$\begin{aligned}\Phi_g(k;\alpha,y) :&= \sum_{m=1}^{\infty} \sqrt{\alpha_g}c_g(m)e(m\alpha)(2\pi my)^{\frac{k}{2}-\frac{1}{4}}e^{-2\pi my}\\ &= \sum_{m=1}^{\infty} \sqrt{\alpha_g}c_g(m)e(m\alpha)I_{k+\frac{1}{2}}(2\pi my).\end{aligned}$$

Let $1 \leq \Delta \ll K$ be a parameter specified later. We shall decompose $\Phi_g(k;\alpha,y)$ into three pieces according to the size of m:

$$\begin{aligned}\Phi(k;\alpha,y) = &\sum_{\substack{m\geq 1\\ 2\pi my<\frac{k}{2}-\frac{1}{4}-\Delta}} \sqrt{\alpha_g}c_g(m)e(m\alpha)I_{k+\frac{1}{2}}(2\pi my) + \sum_{|2\pi my-\frac{k}{2}+\frac{1}{4}|\leq\Delta} \sqrt{\alpha_g}c_g(m)e(m\alpha)I_{k+\frac{1}{2}}(2\pi my)\\ &+ \sum_{2\pi my>\frac{k}{2}-\frac{1}{4}+\Delta} \sqrt{\alpha_g}c_g(m)e(m\alpha)I_{k+\frac{1}{2}}(2\pi my)\\ =:&\ \Phi_g^{(1)}(k;\alpha,y)+\Phi_g^{(2)}(k;\alpha,y)+\Phi_g^{(3)}(k;\alpha,y),\end{aligned}$$

say.

Starting with $\Phi_g^{(2)}(k;\alpha,y)$, we shall use Lemma 6.3. Choosing $h = 2\pi my - (k/2-1/4)$ and $\delta = 1/6-\varepsilon$ (anticipating the choice $\Delta \asymp \sqrt{k\log k}$) we have

$$I_{k+\frac{1}{2}}(2\pi my) = I_{k+\frac{1}{2}}\left(\frac{k}{2}-\frac{1}{4}\right)e^{-|2\pi my-k/2+1/4|^2/(k-1/2)}\left(1+O\left(k^{-3\delta}\right)\right).$$

The error term contributes to $\Phi_g^{(2)}(k;\alpha,y)$ the amount

$$\begin{aligned}&\ll k^{-1/3+\varepsilon}\left(\frac{k}{y}\right)^{1/6+\varepsilon} I_{k+\frac{1}{2}}\left(\frac{k}{2}-\frac{1}{4}\right)\left(\sum_{|2\pi ny-(k/2-1/4)|\leq\Delta} 1\right)k^{-3\delta}\\ &\ll k^{-2/3+\varepsilon}y^{-1/6}I_{k+\frac{1}{2}}\left(\frac{k}{2}-\frac{1}{4}\right)\left(1+\frac{\Delta}{y}\right) \ll k^{-2/3+\varepsilon}y^{-1/6}I_{k+\frac{1}{2}}\left(\frac{k}{2}-\frac{1}{4}\right)\end{aligned}$$

as we shall choose $\Delta \asymp \sqrt{k\log k}$. Our aim now is to show that $\Phi^{(1)}$ and $\Phi^{(3)}$ both give a smaller contribution.

We first concentrate on $\Phi_g^{(3)}$. Using the identity $I_{s_1}(t) = t^{(s_1-s_2)/2} I_{s_2}(t)$ with $s_1 = k+\frac{1}{2}$, $s_2 = k+\frac{5}{6}+2\varepsilon$, and the estimate (6.1) we have

$$\begin{aligned}\Phi_g^{(3)}(g;\alpha,y) &\ll k^{-1/3+\varepsilon} \sum_{\substack{m\geq 1\\ 2\pi my > \frac{k}{2}-\frac{1}{4}+\Delta}} m^{1/6+\varepsilon}(2\pi my)^{-1/6+\varepsilon} I_{k+\frac{5}{6}+2\varepsilon}(2\pi my)\\ &\ll y^{-1/6-\varepsilon}k^{-1/3+\varepsilon} \sum_{\substack{m\geq 1\\ 2\pi my > \frac{k}{2}-\frac{1}{4}+\Delta}} I_{k+\frac{5}{6}+2\varepsilon}(2\pi my).\end{aligned}$$

Note that the function $t \mapsto I_{k+\frac{5}{6}+2\varepsilon}(t)$ achieves its maximum at $t = k/2 - 1/12 + \varepsilon$. On the other hand, as $\Delta \geq 1$, the function $t \mapsto I_{k+\frac{5}{6}+2\varepsilon}(2\pi ty)$ is decreasing in the domain we are considering. Thus we may estimate the sum by an integral as

$$\begin{aligned}\Phi_g^{(3)}(k;\alpha,y) &\ll y^{-1/6-\varepsilon}k^{-1/3+\varepsilon}\left(\int_{(k/2-1/4+\Delta)/2\pi y}^{\infty} (2\pi yt)^{\frac{k}{2}-\frac{1}{12}+\varepsilon}e^{-2\pi yt}\,\mathrm{d}t + I_{k+\frac{5}{6}+2\varepsilon}\left(\frac{k}{2}-\frac{1}{4}+\Delta\right)\right)\\ &\ll y^{-1/6-\varepsilon}k^{-1/3+\varepsilon}\left(\frac{1}{y}\Gamma\left(\frac{k}{2}+\frac{11}{12}+\varepsilon, \frac{k}{2}-\frac{1}{4}+\Delta\right) + I_{k+\frac{5}{6}+2\varepsilon}\left(\frac{k}{2}-\frac{1}{4}+\Delta\right)\right),\end{aligned}$$

where $\Gamma(s,x)$ is the incomplete Gamma function.

By using the Taylor expansion for $\log(1+x)$ we have

$$I_{k+\frac{5}{6}+2\varepsilon}\left(\frac{k}{2}-\frac{1}{4}+\Delta\right) \ll k^{\frac{1}{6}+\varepsilon} I_{k+\frac{1}{2}}\left(\frac{k}{2}-\frac{1}{4}\right) e^{-\Delta^2/(k-1/2)}.$$

Estimating the incomplete Gamma function using a result of Natalini and Palumbo (see [39]) as in [13] we have

$$\begin{aligned}\Gamma\left(\frac{k}{2}+\frac{11}{12}+\varepsilon, \frac{k}{2}-\frac{1}{4}+\Delta\right) &\ll \left(1+\frac{k}{\Delta}+\varepsilon\right)\left(\frac{k}{2}-\frac{1}{4}+\Delta\right)^{\frac{k}{2}-\frac{1}{12}+\varepsilon} e^{-\frac{k}{2}-\frac{1}{4}+\Delta)}\\ &\ll \frac{k}{\Delta}e^{-\Delta^2/(k-1/2)}k^{\frac{1}{6}+\varepsilon} I_{k+\frac{1}{2}}\left(\frac{k}{2}-\frac{1}{4}\right).\end{aligned}$$

Combining all the preceding estimates we have shown that

$$\Phi_g^{(3)}(g;\alpha,y) \ll k^{-1/6+\varepsilon}y^{-1/6-\varepsilon} I_{k+\frac{1}{2}}\left(\frac{k}{2}-\frac{1}{4}\right) e^{-\Delta^2/(k-1/2)}\left(1+\frac{1}{y}\cdot\frac{k}{\Delta}\right).$$

Choosing $\Delta = \sqrt{A(k-1/2)\log k}$ for a sufficiently large fixed constant $A > 0$ the right-hand side of the previous estimate is

$$\ll_\varepsilon k^{-1/6-A+\varepsilon}y^{-1/6} I_{k+\frac{1}{2}}\left(\frac{k}{2}-\frac{1}{4}\right).$$

For $\Phi_g^{(1)}$ note that

$$\sqrt{\alpha_g}c_g(m) \ll k^{-1/3+\varepsilon}m^{1/6+\varepsilon} \ll k^{-1/6+\varepsilon}y^{-1/6-\varepsilon}.$$

In addition, the function $t \mapsto I_{k+\frac{1}{2}}(2\pi ty)$ is strictly increasing in the interval we are considering, and so we may again approximate the sum by an interval as

(6.2)

$$\Phi_g^{(1)}(k;\alpha,y) \ll k^{-1/6+\varepsilon}y^{-1/6-\varepsilon}\left(\int_1^{(k/2-1/4-\Delta)/2\pi y} (2\pi yt)^{\frac{k}{2}-\frac{1}{4}}e^{-2\pi yt}\,\mathrm{d}t + I_{k+\frac{1}{2}}(2\pi y) + I_{k+\frac{1}{2}}\left(\frac{k}{2}-\frac{1}{4}-\Delta\right)\right).$$

Now by the definition of $I_s(y)$ we have

$$I_{k+\frac{1}{2}}(2\pi y) \ll I_{k+\frac{1}{2}}\left(\frac{k}{2}-\frac{1}{4}\right)\left(\frac{2\pi ye}{k/2-1/4}\right)^{\frac{k}{2}-\frac{1}{4}} e^{-2\pi y},$$

which decays exponentially e.g. when $y < \frac{1}{100}k$.

As above, using the Taylor series approximation of $\log(1-x)$ we have

$$I_{k+\frac{1}{2}}\left(\frac{k}{2}-\frac{1}{4}-\Delta\right) \ll I_{k+\frac{1}{2}}\left(\frac{k}{2}-\frac{1}{4}\right) e^{-\Delta^2/(k-1/2)}.$$

The integral in (6.2) is estimated by splitting it into two parts. Let $\Delta_1 > \Delta$ be a parameter specified later, and write

$$\int_1^{(k/2-1/4+\Delta)/2\pi y} (2\pi yt)^{\frac{k}{2}-\frac{1}{4}} e^{-2\pi yt}\,\mathrm{d}t = \left(\int_1^{(k/2-1/4-\Delta_1)/2\pi y} + \int_{(k/2-1/4-\Delta_1)/2\pi y}^{(k/2-1/4-\Delta)/2\pi y}\right) (2\pi yt)^{\frac{k}{2}-\frac{1}{4}} e^{-2\pi yt}\,\mathrm{d}t.$$

By making the change of variables $t \mapsto (\frac{k}{2}-\frac{1}{4})t/2\pi y$ this simplifies into

$$\begin{aligned}
&\frac{\left(\frac{k}{2}-\frac{1}{4}\right)}{2\pi y}\left(\int_{2\pi y/(k/2-1/4)}^{1-\Delta_1/(k/2-1/4)} + \int_{1-\Delta_1/(k/2-1/4)}^{1-\Delta/(k/2-1/4)}\right) t^{\frac{k}{2}-\frac{1}{4}} e^{-\left(\frac{k}{2}-\frac{1}{4}\right)t}\,\mathrm{d}t \\
&= \frac{\left(\frac{k}{2}-\frac{1}{4}\right)}{2\pi y} I_{k+\frac{1}{2}}\left(\frac{k}{2}-\frac{1}{4}\right)\left(\int_{2\pi y/(k/2-1/4)}^{1-\Delta_1/(k/2-1/4)} + \int_{1-\Delta_1/(k/2-1/4)}^{1-\Delta/(k/2-1/4)}\right) t^{\frac{k}{2}-\frac{1}{4}} e^{\left(\frac{k}{2}-\frac{1}{4}\right)(1-t)}\,\mathrm{d}t.
\end{aligned}$$

As the integrand is strictly increasing, the latter integral is

$$\ll \frac{\Delta_1-\Delta}{k}\left(1-\frac{\Delta}{k/2-1/4}\right)^{\frac{k}{2}-\frac{1}{4}} e^{\Delta} \ll \frac{\Delta_1-\Delta}{k} e^{-\Delta^2/(k-1/2)},$$

where we have again used the Taylor expansion of $\log(1-x)$ in the latter estimate.

Similarly, estimating by absolute values and using the Taylor expansion the other part of the integral is $\ll e^{-\Delta_1^2/(k-1/2)}$ provided that $\Delta_1 = o(k^{2/3})$, which will be satisfied as we shall choose $\Delta_1 := \sqrt{B(k-1/2)\log k}$ for a fixed constant $B > A + \frac{1}{2}$. With this choice, gathering all the estimates, we have

$$\begin{aligned}
\Phi_g^{(1)}(k;\alpha,y) &\ll k^{-1/6+\varepsilon} y^{-1/6-\varepsilon} I_{k+\frac{1}{2}}\left(\frac{k}{2}-\frac{1}{4}\right)\frac{k}{y}\left(\frac{\Delta_1-\Delta}{k/2-1/4} e^{-\Delta^2/(k-1/2)} + e^{-\Delta_1^2/(k-1/2)}\right) \\
&\ll k^{-1/6+\varepsilon} y^{-1/6-\varepsilon} I_{k+\frac{1}{2}}\left(\frac{k}{2}-\frac{1}{4}\right)\frac{k}{y}\left(\frac{\sqrt{k\log k}}{k} k^{-A} + k^{-B}\right) \\
&\ll k^{-1/6-A+\varepsilon} y^{-1/6-\varepsilon} I_{k+\frac{1}{2}}\left(\frac{k}{2}-\frac{1}{4}\right),
\end{aligned}$$

where the last step follows from recalling that $B > A + 1/2$ and choosing A to be sufficiently large.

In total we have shown that

$$\begin{aligned}
&\sqrt{\alpha_g}\, g(\alpha+iy) \\
&= (2\pi y)^{-\frac{k}{2}+\frac{1}{4}} I_{k+\frac{1}{2}}\left(\frac{k}{2}-\frac{1}{4}\right)\Bigg(\sum_{\substack{m\geq 1 \\ |2\pi my-(k/2-1/4)|\leq\Delta}} \sqrt{\alpha_g}\, c_g(m) e(m\alpha) e^{-|2\pi my-(k/2-1/4)|^2/(k-1/2)} \\
&\qquad\qquad + O\left(k^{-2/3+\varepsilon} y^{-1/6}\right)\Bigg).
\end{aligned}$$

Now observe that for $y_\ell = (k-1/2)/4\pi\ell$,

$$\left|2\pi m y_\ell - \left(\frac{k}{2}-\frac{1}{4}\right)\right| \leq \Delta \quad \Longleftrightarrow \quad |m-\ell| \leq \frac{\Delta\ell}{k/2-1/4},$$

which forces $m=\ell$ in the light of the choice for Δ when $\ell \leq \sqrt{(k/2-1/4)/(2A\log k)}$.

Hence,

$$\begin{aligned}\sqrt{\alpha_g} g(\alpha+iy_\ell) &= (2\pi y_\ell)^{-\frac{k}{2}+\frac{1}{4}} I_{k+\frac{1}{2}}\left(\frac{k}{2}-\frac{1}{4}\right)\left(\sqrt{\alpha_g} c_g(\ell) e(\alpha\ell) + O\left(k^{-1/2-\eta}\right)\right) \\ &= \left(\frac{2\ell}{e}\right)^{\frac{k}{2}-\frac{1}{4}}\left(\sqrt{\alpha_g} c_g(\ell) e(\alpha\ell) + O\left(k^{-1/2-\eta}\right)\right)\end{aligned}$$

for, say, $\eta = 1/3+\varepsilon$. This completes the proof. □

7. Character sum

In the proof of Lemma 2.8 we require the evaluation of a certain character sum. In this section we achieve this task.

Lemma 7.1. *Let d be an odd squarefree integer and let $c, v,$ and η be natural numbers so that $v\eta = [c,d]^2/c^2$. Then the sum*

$$\sum_{x\,([c,d])} \sum_{w\,([c,d])} \chi_d(x)\chi_d(w) S(x,w;c)\, e\left(\frac{xv+w\eta}{[c,d]}\right)$$

vanishes unless[12] *$d|c$, in which case it equals $c^2\varphi(d)/d$.*

Proof. After opening the Kloosterman sum and rearranging our task is to evaluate

$$\sideset{}{^\star}\sum_{\gamma\,(c)} \left(\sum_{x\,([c,d])} \chi_d(x) e\left(\frac{x\gamma}{c}+\frac{xv}{[c,d]}\right)\right)\left(\sum_{w\,([c,d])} \chi_d(w) e\left(\frac{w\overline{\gamma}}{c}+\frac{w\eta}{[c,d]}\right)\right). \tag{7.1}$$

Let us focus on the second factor. Using the formula $(c,d)\cdot[c,d] = cd$ it may be written in the form

$$\sum_{w\,([c,d])} \chi_d(w) e\left(\frac{w}{[c,d]}\left(\frac{d\overline{\gamma}}{(c,d)}+\eta\right)\right).$$

Just to simplify notation, define

$$\beta_1 := \frac{d\overline{\gamma}}{(c,d)}+\eta.$$

We split the sum into congruence classes modulo d and change the order of summation to get

$$\begin{aligned}&\sum_{k=0}^{[c,d]/d-1} \sum_{y\,(d)} \chi_d(y+kd) e\left(\frac{(y+kd)\beta_1}{[c,d]}\right) \\ &= \sum_{y\,(d)} \chi_d(y) e\left(\frac{y\beta_1}{[c,d]}\right) \sum_{k=0}^{[c,d]/d-1} e\left(\frac{k\beta_1}{[c,d]/d}\right).\end{aligned}$$

By orthogonality the inner sum in the last display equals

$$\begin{cases} \frac{[c,d]}{d} & \text{if } \beta_1 \equiv 0\,([c,d]/d) \\ 0 & \text{otherwise} \end{cases}$$

We write $\beta_2 := d\beta_1/[c,d]$. With this notation the sum equals

$$\frac{[c,d]}{d} \cdot 1_{\beta_1 \equiv 0\,([c,d]/d)} \sum_{y\,(d)} \chi_d(y) e\left(\frac{y\beta_2}{d}\right).$$

[12]Note also that the condition $d|c$ forces $v = \eta = 1$.

Note that if we write ι for (β_2, d) and $\beta_2 = \iota \cdot \beta_3$ and $d = \iota \cdot d_1$ with $(\beta_3, d_1) = 1$, it follows that $(d_1, \iota) = 1$ as d is squarefree. Note that by the Chinese Remainder Theorem

$$\sum_{y\,(d)} \chi_d(y) e\left(\frac{y\beta_2}{d}\right) = \sum_{y_1\,(d_1)} \chi_{d_1}(y_1) e\left(\frac{y_1\beta_3}{d_1}\right) \sum_{y_2\,(\iota)} \chi_\iota(y_2),$$

with the inner sum vanishing unless $\iota = 1$. In this case the outer sum is recognised as a quadratic Gauss sum with the value $\sqrt{d}\chi_d(\beta_2)$ as d was assumed to be odd and squarefree. In conclusion, the second factor in (7.1) equals

$$1_{\beta_1 \equiv 0\,([c,d]/d)} \cdot \frac{[c,d]}{\sqrt{d}} \chi_d(\beta_2).$$

Completely analogous reasoning gives that the first factor equals

$$1_{\delta_1 \equiv 0\,([c,d]/d)} \cdot \frac{[c,d]}{\sqrt{d}} \chi_d(\delta_2),$$

where

$$\delta_1 := \frac{d\gamma}{(c,d)} + v$$

and $\delta_2 := d\delta/[c,d]$.

Let us write $c = c' \cdot (c,d)$ and $d = d' \cdot (c,d)$ with $(c', d') = 1$. Note that

$$\beta_2 = \frac{d'\overline{\gamma} + \eta}{c'} \qquad \text{and} \qquad \delta_2 = \frac{\gamma d' + v}{c'}.$$

Recall that by assumption, $v\eta = [c,d]^2/c^2 = d'^2$. In particular, if $d' \neq 1$, there exists a prime p dividing d' and consequently also $v\eta$. Thus at least one of the terms $\overline{\gamma}d' + \eta$ and $\gamma d' + v$ is divisible by p. As $(c', p) = 1$ due to $(c', d') = 1$ it follows that $\chi_d(\beta_2)\chi_d(\delta_2) = 0$ independent of γ and so the original sum vanishes. Note that $d' = 1$ if and only if $d = (d, c)$ i.e. $d|c$ and so we conclude that the sum vanishes unless $d|c$.

In this case our sum is given by

$$\frac{c^2}{d} \sideset{}{^*}\sum_{\gamma\,(c)} \chi_d\left(\frac{\overline{\gamma} + \eta}{c'}\right) \chi_d\left(\frac{\gamma + v}{c'}\right) \cdot 1_{\beta_1 \equiv 0\,(c/d)} \cdot 1_{\delta_1 \equiv 0\,(c/d)}.$$

The congruence conditions $\beta_1, \delta_1 \equiv 0\,(d)$ are equivalent to $v + \gamma \equiv 0\,(c')$, where now $c' = c/d$. Also as $v\eta = 1$ we have

$$\chi_d\left(\frac{\overline{\gamma} + \eta}{c'}\right) \chi_d\left(\frac{\gamma + v}{c'}\right) = \chi_d\left(\frac{\overline{\gamma} + 1}{c'}\right) \chi_d\left(\frac{\gamma + 1}{c'}\right).$$

Note also that

$$(\overline{\gamma} + 1)(\gamma + 1) \equiv \gamma(\overline{\gamma} + 1)^2 \pmod{d}$$

and so

$$\chi_d\left(\frac{\overline{\gamma} + 1}{c'}\right) \chi_d\left(\frac{\gamma + 1}{c'}\right) = \chi_d\left(\frac{\gamma(\overline{\gamma} + 1)^2}{(c')^2}\right)$$

As $1 + \gamma \equiv 0\,(c')$ if and only if $\overline{\gamma} + 1 \equiv 0\,(c')$ (recall that $(\overline{\gamma}, c') = 1$) it follows that our sum is at this point given by

$$\frac{c^2}{d} \sideset{}{^*}\sum_{\substack{\gamma\,(c) \\ \gamma + 1 \equiv 0\,(c') \\ \left(\frac{\overline{\gamma}+1}{c'}, d\right) = 1}} \chi_d(\gamma).$$

But as $\chi_{c'}(\gamma) = 1$ for $\gamma \equiv -1\,(c')$, it follows that the sum equals

$$\begin{aligned}\frac{c^2}{d} \sideset{}{^*}\sum_{\substack{\gamma\,(c)\\ \gamma+1\equiv 0\,(c')\\ \left(\frac{\overline{\gamma}+1}{c'},d\right)=1}} \chi_d(\gamma)\chi_{c'}(\gamma) &= \frac{c^2}{d} \sideset{}{^*}\sum_{\substack{\gamma\,(c)\\ \gamma+1\equiv 0\,(c')\\ \left(\frac{\overline{\gamma}+1}{c'},d\right)=1}} \chi_c(\gamma)\\ &= \frac{c^2}{d} \sideset{}{^*}\sum_{\substack{\gamma\,(c)\\ \gamma+1\equiv 0\,(c')}} \chi_c(\gamma) \sum_{j\mid\left(\frac{\gamma+1}{c'},d\right)} \mu(j)\\ &= \frac{c^2}{d}\sum_{j\mid d}\mu(j) \sideset{}{^*}\sum_{\substack{\gamma\,(c)\\ 1+\gamma\equiv 0\,(c'j)}} \chi_c(\gamma),\end{aligned}$$

where the penultimate step follows by Möbius inversion.

Using [23, Lemma 5.2] the inner sum equals $c/c'j = d/j$. Collecting all the above computations shows that our sum equals, when $d|c$,

$$\frac{c^2}{d}\cdot d\sum_{j|d}\frac{\mu(j)}{j} = \frac{c^2\varphi(d)}{d},$$

as desired. This completes the proof. □

8. Proof of Lemma 2.7

In this section we evaluate the average of the absolute squares of Fourier coefficients of half-integral weight Hecke cusp forms.

8.1. Proof of Lemma 2.7. Recall that the sum we aim to evaluate is

$$S_1 := \sum_{k\in\mathbb{Z}} h\left(\frac{2k-1}{K}\right) \sum_{g\in B^+_{k+\frac{1}{2}}} \alpha_g \sideset{}{^\flat}\sum_{d} |c_g(|d|)|^2 \phi\left(\frac{(-1)^k d}{X}\right).$$

We begin by detecting the condition that d is squarefree by means of the identity

$$\sum_{\substack{\alpha=1\\ \alpha^2|d}}^{\infty} \mu(\alpha) = \begin{cases} 1 & d \text{ is squarefree}\\ 0 & \text{otherwise}\end{cases} \tag{8.1}$$

This together with the relation (2.5) and the approximate functional equation shows that the sum in question equals

$$\begin{aligned} S_1 &= 2\sum_{k\in\mathbb{Z}} h\left(\frac{2k-1}{K}\right)\sum_{f\in\mathcal{B}_k}\omega_f \sum_{d\equiv 1\,(16)} \sum_{\substack{\alpha=1\\ \alpha^2|d}}^{\infty}\mu(\alpha)\sum_{m=1}^{\infty}\frac{\lambda_f(m)\chi_d(m)}{\sqrt{m}} V_k\left(\frac{m}{|d|}\right)\phi\left(\frac{(-1)^k d}{X}\right)\\ &= 2\sum_{k\in\mathbb{Z}} h\left(\frac{2k-1}{K}\right) \sum_{\substack{\alpha=1\\ (\alpha,2)=1}}^{\infty}\mu(\alpha) \sum_{d\equiv\overline{\alpha^2}\,(16)}\sum_{f\in\mathcal{B}_k}\omega_f \sum_{\substack{m=1\\ (m,\alpha)=1}}^{\infty} \frac{\lambda_f(m)\chi_d(m)}{\sqrt{m}}\phi\left(\frac{(-1)^k\alpha^2 d}{X}\right) V_k\left(\frac{m}{\alpha^2|d|}\right). \end{aligned}\tag{8.2}$$

Let us define $F(\xi;x,y) := \phi(\xi/x)V_k(y/\xi)$. With this definition we have

$$\begin{aligned} F\left(d;\frac{X}{(-1)^k\alpha^2},\frac{m}{(-1)^k\alpha^2}\right) &= \phi\left(\frac{(-1)^k d\alpha^2}{X}\right)V_k\left(\frac{m}{(-1)^k d\alpha^2}\right)\\ &= \phi\left(\frac{(-1)^k\alpha^2 d}{X}\right)V_k\left(\frac{m}{\alpha^2|d|}\right)\end{aligned}$$

as the support of ϕ is contained in the positive real numbers.

Separating the α-sum into two parts depending on whether $\alpha \le Z$ (for some parameter $1 \le Z \le X$ chosen later) or not gives

$$S_1 = 2\sum_{k\in\mathbb{Z}} h\left(\frac{2k-1}{K}\right) \sum_{f\in\mathcal{B}_k} \omega_f \left(\sum_{\substack{\alpha\le Z\\(\alpha,2)=1}} + \sum_{\substack{\alpha>Z\\(\alpha,2)=1}} \right) \mu(\alpha) \times \sum_{\substack{m=1\\(m,\alpha)=1}}^{\infty} \frac{\lambda_f(m)}{\sqrt{m}} \sum_{d\equiv\overline{\alpha^2}\,(16)} \chi_d(m) F\left(d; \frac{(-1)^k X}{\alpha^2}, \frac{(-1)^k m}{\alpha^2}\right).$$

First we treat the contribution of the terms with $\alpha > Z$. Arguing like in [20], but using the estimate

$$\sum_{f\in\mathcal{B}_k} \omega_f L\left(\frac{1}{2}, f\otimes\chi_d\right) \ll_\varepsilon (|d|k)^\varepsilon,$$

which holds uniformly in $|d| \le k^2$ (see [35, (1.11.)]), in place of the Generalised Lindelöf Hypothesis for $L(1/2, f\otimes\chi_d)$ shows that

$$\left| \sum_{k\in\mathbb{Z}} h\left(\frac{2k-1}{K}\right) \sum_{f\in\mathcal{B}_k} \omega_f \sum_{\substack{\alpha>Z\\(\alpha,2)=1}} \sum_{\substack{m=1\\(m,\alpha)=1}}^{\infty} \frac{\lambda_f(m)}{\sqrt{m}} \sum_{d\equiv\overline{\alpha^2}\,(16)} \mu(\alpha)\chi_d(m) F\left(d; \frac{(-1)^k X}{\alpha^2}, \frac{(-1)^k m}{\alpha^2}\right) \right| \ll_\varepsilon (KX)^\varepsilon \frac{XK}{Z}.$$

Next we treat the part corresponding to the terms with $\alpha \le Z$. We begin by applying the Petersson formula. This gives

$$2\sum_{k\in\mathbb{Z}} h\left(\frac{2k-1}{K}\right) \sum_{\substack{\alpha\le Z\\(\alpha,2)=1}} \mu(\alpha) \sum_{d\equiv\overline{\alpha^2}\,(16)} \sum_{\substack{m=1\\(m,\alpha)=1}}^{\infty} \frac{\chi_d(m)}{\sqrt{m}} F\left(d; \frac{(-1)^k X}{\alpha^2}, \frac{(-1)^k m}{\alpha^2}\right) \times \left(1_{m=1} + 2\pi i^{2k} \sum_{c=1}^{\infty} \frac{S(m,1;c)}{c} J_{2k-1}\left(\frac{4\pi\sqrt{m}}{c}\right)\right).$$

The first term in the parenthesis gives the diagonal contribution

$$2\sum_{k\in\mathbb{Z}} h\left(\frac{2k-1}{K}\right) \sum_{\substack{\alpha\le Z\\(\alpha,2)=1}} \mu(\alpha) \sum_{d\equiv\overline{\alpha^2}\,(16)} F\left(d; \frac{(-1)^k X}{\alpha^2}, \frac{(-1)^k}{\alpha^2}\right).$$

The idea is to evaluate the d-sum using Lemma 5.7. This yields

$$\frac{1}{8}\sum_{k\in\mathbb{Z}} h\left(\frac{2k-1}{K}\right) \sum_{\substack{\alpha\le Z\\(\alpha,2)=1}} \mu(\alpha) \sum_{\ell\in\mathbb{Z}} \widehat{F}\left(\frac{\ell}{16}; \frac{(-1)^k X}{\alpha^2}, \frac{(-1)^k}{\alpha^2}\right) e\left(\frac{\ell\overline{\alpha^2}}{16}\right) \tau_\ell(1). \tag{8.3}$$

The terms with $\ell \ne 0$ are easily estimated using the decay properties of $\widehat{F}$. First we have the estimate

$$\widehat{F}(\lambda; y, z) \ll_{\phi,A} y \min\left\{ \left(\frac{Ky}{z}\right)^A, \left(\frac{K}{|z|\lambda}\right)^A \right\} \tag{8.4}$$

for any integer $A > 0$ (cf. [20, (5.6)]). From this it follows that $\widehat{F}(\ell/16; (-1)^k X/\alpha^2, (-1)^k/\alpha^2)$ is negligible if $|\ell| \ge 16\alpha^2 K(XK)^\varepsilon$. Moreover, write $\Phi(s) := \int_0^\infty \phi(x)x^s\,\mathrm{d}x$ and let

$$\widetilde{F}(s,\ell,\alpha^2) := \int_0^\infty \widehat{F}\left(\frac{\ell}{16t}; \frac{(-1)^k X}{\alpha^2}, \frac{(-1)^k t}{\alpha^2}\right) t^{s-1}\,\mathrm{d}t = \frac{X^{1+s}}{\alpha^2}\Phi(s) \int_0^\infty V_k\left(\frac{1}{y}\right) e\left(\frac{-\ell y}{16\alpha^2}\right) \frac{\mathrm{d}y}{y^{s+1}}. \tag{8.5}$$

Recall that $V_k(\xi) = 1 + O_\varepsilon(\xi^{1/2-\varepsilon})$ as $\xi \longrightarrow 0$. Using this we see that the function $\widetilde{F}$ admits an analytic continuation to $\mathrm{Re}(s) \geq -1$ and furthermore for $-1+\varepsilon \leq \mathrm{Re}(s) \leq 2$, any non-negative integer A, and $\ell \in \mathbb{Z} \setminus \{0\}$ that

$$\widetilde{F}(s, \ell, \alpha^2) \ll_{\phi, A} \frac{X^{1+\mathrm{Re}(s)} K^{\mathrm{Re}(s)}}{\alpha^2} \left(\left(\frac{|\ell|}{\alpha^2 K} \right)^{\mathrm{Re}(s)} + 1 \right) \left(\frac{1}{1+|s|} \right)^A. \tag{8.6}$$

Define the function

$$\mathcal{F}(t) := \widehat{F}\left(\frac{\ell}{16t}; \frac{(-1)^k X}{\alpha^2}, \frac{(-1)^k t}{\alpha^2} \right).$$

By Mellin inversion we have

$$\widehat{F}\left(\frac{\ell}{16}; \frac{(-1)^k X}{\alpha^2}, \frac{(-1)^k}{\alpha^2} \right) = \mathcal{F}(1) = \frac{1}{2\pi i} \int_{(2)} \widetilde{F}(s, \ell, \alpha^2)\, \mathrm{d}s.$$

Shifting the line of integration to the line $\sigma = -1 + \varepsilon$ and using (8.6) shows that

$$\widehat{F}\left(\frac{\ell}{16}; \frac{(-1)^k X}{\alpha^2}, \frac{(-1)^k}{\alpha^2} \right) \ll_\varepsilon \frac{K^\varepsilon}{\alpha^2} \left(\frac{\alpha^2}{|\ell|} \right)^{1-\varepsilon}.$$

Using this and estimating trivially the terms with $\ell \neq 0$ contribute an amount

$$\ll_\varepsilon K^{1+\varepsilon} Z.$$

to (8.3).

The main contribution will come from the part corresponding to the summand with $\ell = 0$. As $\tau_0(1) = 1$ this part equals

$$\frac{1}{8} \sum_{k \in \mathbb{Z}} h\left(\frac{2k-1}{K} \right) \sum_{\substack{\alpha=1 \\ (\alpha,2)=1}}^{\infty} \mu(\alpha) \widehat{F}\left(0; \frac{(-1)^k X}{\alpha^2}, \frac{(-1)^k}{\alpha^2} \right) + O\left(\frac{X K^{1+\varepsilon}}{Z} \right) \tag{8.7}$$

by adding back the contribution $\alpha > Z$ using (8.4).

By definition we have

$$\widehat{F}\left(0; \frac{(-1)^k X}{\alpha^2}, \frac{(-1)^k}{\alpha^2} \right) = \frac{X}{\alpha^2} \cdot \frac{1}{2\pi i} \int_{(\sigma)} \frac{\Gamma(s+k)}{\Gamma(s)} \left(\frac{X}{2\pi} \right)^s e^{s^2} \left(\int_{\mathbb{R}} \phi(\xi) \xi^s \, \mathrm{d}\xi \right) \frac{\mathrm{d}s}{s}.$$

Substituting this into (8.7) gives the main term

$$\frac{X}{8} \sum_{k \in \mathbb{Z}} h\left(\frac{2k-1}{K} \right) \sum_{\substack{\alpha=1 \\ (\alpha,2)=1}}^{\infty} \frac{\mu(\alpha)}{\alpha^2} \cdot \frac{1}{2\pi i} \int_{(\sigma)} \left(\int_{\mathbb{R}} \phi(\xi) \xi^s \, \mathrm{d}\xi \right) \frac{\Gamma(s+k)}{\Gamma(k)} \left(\frac{X}{2\pi} \right)^s e^{s^2} \frac{\mathrm{d}s}{s} + O\left(\frac{X K^{1+\varepsilon}}{Z} \right).$$

We note that

$$\sum_{\substack{\alpha=1 \\ (\alpha,2)=1}}^{\infty} \frac{\mu(\alpha)}{\alpha^2} = \frac{8}{\pi^2}.$$

Furthermore, the integrand has a simple pole at $s = 0$ with residue $\widehat{\phi}(0)$. Hence, moving the line of integration to $\mathrm{Re}(s) = -1/2 + \varepsilon$ yields the main term

$$\frac{XK}{2\pi^2} \widehat{h}(0) \widehat{\phi}(0) \tag{8.8}$$

by summing over k using the Poisson summation (up to a negligible error). On the new line the integral is $\ll X^{-1/2+\varepsilon}$ and so combining the estimates above we have shown that the diagonal contribution equals

$$\frac{XK}{2\pi^2} \widehat{h}(0) \widehat{\phi}(0) + O\left(K^\varepsilon \left(\frac{XK}{Z} + KZ \right) \right).$$

We now move to off-diagonal terms, which are given by

$$4\pi\sum_{k\in\mathbb{Z}}h\left(\frac{2k-1}{K}\right)\sum_{\substack{\alpha\le Z\\(\alpha,2)=1}}\mu(\alpha)\sum_{d\equiv\alpha^2\,(16)}\sum_{\substack{m=1\\(m,\alpha)=1}}^{\infty}\frac{\chi_d(m)}{\sqrt{m}}F\left(d;\frac{(-1)^kX}{\alpha^2},\frac{(-1)^km}{\alpha^2}\right) \times i^{2k}\sum_{c=1}^{\infty}\frac{S(m,1;c)}{c}J_{2k-1}\left(\frac{4\pi\sqrt{m}}{c}\right). \tag{8.9}$$

At this point we separate even and odd parts in the m-variable. As $\chi_d(2^j)=1$ for $d\equiv 1\,(16)$, the sum (8.9) can be written in the form

$$4\pi\sum_{k\in\mathbb{Z}}h\left(\frac{2k-1}{K}\right)\sum_{\substack{\alpha\le Y\\(\alpha,2)=1}}\mu(\alpha)\sum_{d\equiv\alpha^2\,(16)}\sum_{j=0}^{\infty}\sum_{\substack{m=1\\(m,2\alpha)=1}}^{\infty}\frac{\chi_d(m)}{\sqrt{2^jm}}F\left(d;\frac{(-1)^kX}{\alpha^2},\frac{(-1)^k2^jm}{\alpha^2}\right) \times i^{2k}\sum_{c=1}^{\infty}\frac{S(2^jm,1;c)}{c}J_{2k-1}\left(\frac{4\pi\sqrt{2^jm}}{c}\right).$$

We again evaluate the d-sum using Lemma 5.7. This transforms the previous display into

$$\frac{\pi}{4}\sum_{k\in\mathbb{Z}}h\left(\frac{2k-1}{K}\right)\sum_{\substack{\alpha\le Z\\(\alpha,2)=1}}\mu(\alpha) \times\sum_{j=0}^{\infty}\frac{1}{2^{j/2}}\sum_{\substack{m=1\\(m,2\alpha)=1}}^{\infty}\frac{1}{m^{3/2}}i^{2k}\sum_{c=1}^{\infty}\frac{S(2^jm,1;c)}{c}J_{2k-1}\left(\frac{4\pi\sqrt{2^jm}}{c}\right) \times\sum_{\ell\in\mathbb{Z}}\widehat{F}\left(\frac{\ell}{16m},\frac{(-1)^kX}{\alpha^2},\frac{(-1)^k2^jm}{\alpha^2}\right)e\left(\frac{\ell\overline{\alpha^2m}}{16}\right)\tau_\ell(m). \tag{8.10}$$

We again treat the parts corresponding to $\ell=0$ and $\ell\neq 0$ separately. The terms with $\ell=0$ contribute

$$\frac{\pi}{4}\sum_{k\in\mathbb{Z}}i^{2k}h\left(\frac{2k-1}{K}\right)\sum_{\substack{\alpha\le Z\\(\alpha,2)=1}}\mu(\alpha)\sum_{j=0}^{\infty}\frac{1}{2^{j/2}}\sum_{\substack{m=1\\(m,2\alpha)=1}}^{\infty}\frac{\tau_0(m)}{m^{3/2}} \times\sum_{c=1}^{\infty}\frac{S(m,2^j;c)}{c}\widehat{F}\left(0,\frac{(-1)^kX}{\alpha^2},\frac{(-1)^k2^jm}{\alpha^2}\right)J_{2k-1}\left(\frac{4\pi\sqrt{2^jm}}{c}\right).$$

Executing the k-sum using (5.11) yields, up to an error $O(X/K^2)$,

$$-\frac{\sqrt{\pi}K}{8}\sum_{\substack{\alpha\le Z\\(\alpha,2)=1}}\mu(\alpha)\sum_{j=0}\frac{1}{2^{3j/4}} \times\sum_{\substack{m=1\\(m,\alpha)=1}}^{\infty}\frac{\tau_0(m)}{m^{7/4}}\sum_{c=1}^{\infty}\frac{S(m,2^j;c)}{\sqrt{c}}\cdot\mathrm{Im}\left(e^{-2\pi i/8}e\left(\frac{2\sqrt{2^jm}}{c}\right)\int_{\mathbb{R}}\phi\left(\frac{\xi\alpha^2}{X}\right)W_K^{(1)}\left(\frac{2^jm}{\alpha^2\xi},\frac{K^2c}{8\pi\sqrt{2^jm}}\right)\mathrm{d}\xi\right).$$

By (5.10) we may restrict the sums to $m\le K^{1+\varepsilon}X$ and $c\le\sqrt{m}/K^{2-\varepsilon}$ with an error of $O\left(K^{-10}\right)$. Just an application of Weil's bound gives that the previous display is

$$\ll X^{7/4}K^{-1/4+\varepsilon},$$

which is certainly $O(X^{1/2}K^{1+\varepsilon})$ as $X\ll\sqrt{K}$.

It remains to estimate the contribution from the terms with $\ell\neq 0$ in (8.10). We split our estimate into two cases depending on whether $|\ell|\ge L$ where $L:=16\alpha^2K(XK)^\varepsilon$. Using (8.4) we have $\widehat{F}(\ell/16m,X/\alpha^2,2^jm/\alpha^2)$

decays rapidly when $|\ell| \geq L$ and adapting the argument given in [41, Section 10.3] we get that the contribution of the terms with with $|\ell| \geq L$ to the right-hand side of (8.10) is $\ll (KX)^{-100}$. Finally, we consider the terms with $0 < |\ell| < L$. The argument is similar to the one given in [20], but we use the Lindelöf bound on average in the weight-aspect for twisted L-functions in place of GLH. First we express the additive character $e(\ell\overline{\alpha^2 m}/16)$ in terms of Dirichlet characters modulo 16, using orthogonality of characters as in [41, p. 1065], to see that these terms are bounded by

$$\ll \sum_{0<|\ell|<L} \sum_{\alpha \leq Z} \sum_{\psi\,(16)} \Bigg| \sum_{k\in\mathbb{Z}} h\left(\frac{2k-1}{K}\right) i^{2k} \sum_{j=0}^{\infty} \frac{1}{2^{j/2}} \sum_{\substack{m=1\\(m,2\alpha)=1}}^{\infty} \frac{\tau_\ell(m)\psi(m)}{m^{3/2}} \tag{8.11}$$

$$\times \sum_{c=1}^{\infty} \frac{S(2^j m,1;c)}{c} J_{2k-1}\left(\frac{4\pi\sqrt{2^j m}}{c}\right) \widehat{F}\left(\frac{\ell}{16m}; \frac{X}{\alpha^2}, \frac{2^j m}{\alpha^2}\right) \Bigg|. \tag{8.12}$$

Applying Mellin inversion, the sum inside the absolute values in (8.11) is

$$\sum_{k\in\mathbb{Z}} h\left(\frac{2k-1}{K}\right) \frac{1}{2\pi i} \int_{(2)} \widetilde{F}(s,\ell,\alpha^2) \left(\sum_{j=0}^{\infty} \frac{1}{2^{j/2}} \sum_{\substack{m=1\\(m,2\alpha)=1}}^{\infty} \frac{\tau_\ell(m)\psi(m)}{m^{3/2+s}} \sum_{c=1}^{\infty} \frac{S(2^j m,1;c)}{c} J_{2k-1}\left(\frac{4\pi\sqrt{2^j m}}{c}\right)\right) \mathrm{d}s. \tag{8.13}$$

We execute the k-sum, which transforms this into

$$-K \sum_{j=0}^{\infty} \frac{1}{2^{j/2}} \sum_{\substack{m=1\\(m,2\alpha)=1}}^{\infty} \frac{\tau_\ell(m)\psi(m)}{m^{7/4+s}} \sum_{c=1}^{\infty} \frac{S(2^j m,1;c)}{\sqrt{c}} e\left(\frac{2\sqrt{2^j m}}{c}\right)$$
$$\times \frac{1}{2\pi i} \int_{(2)} \frac{X^{1+s}}{\alpha^2} \Phi(s) \left(\int_0^{\infty} W_K^{(1)}\left(\frac{1}{y}, \frac{K^2 c}{8\pi\sqrt{2^j m}}\right) e\left(-\frac{\ell y}{16\alpha^2}\right) \frac{\mathrm{d}y}{y^{s+1}} \right) \mathrm{d}s.$$

Shifting the line of integration in the s-integral to $\sigma = -1+\varepsilon$ and estimating trivially with the Weil bound shows that the contribution of the terms $0 < |\ell| \leq L$ to (8.11) is again $\ll_\varepsilon K^{1+\varepsilon} Z$.

Combining this with (8.7) and (8.8) shows that the sum S_1 equals

$$\frac{XK}{2\pi^2} \widehat{h}(0)\widehat{\phi}(0) + O_{h,\phi,\varepsilon}\left((XK)^\varepsilon \left(\frac{XK}{Z} + KZ\right)\right).$$

To balance error terms we take $Z = X^{1/2}$, which completes the proof. □

9. Proof of Lemma 2.8

We now move into the proof of the second lemma. Our argument borrows ingredients from both [33] and [22].

Recall that the sum we are considering is

$$S_2 := \sum_{k\in\mathbb{Z}} h\left(\frac{2k-1}{K}\right) \sum_{g\in B^+_{k+\frac{1}{2}}} \alpha_g^2 \omega_g^{-1} \sideset{}{^\flat}\sum_{d} |c_g(|d|)|^4 \phi\left(\frac{(-1)^k d}{X}\right).$$

Using the relation (2.5) our sum takes the form

$$\sum_{k\in\mathbb{Z}} h\left(\frac{2k-1}{K}\right) \sum_{f\in\mathcal{B}_k} \sideset{}{^\flat}\sum_{d} \omega_f L\left(\frac{1}{2}, f\otimes\chi_d\right)^2 \phi\left(\frac{(-1)^k d}{X}\right),$$

and applying the approximate functional equation transforms this further into

$$4\sum_{k\in\mathbb{Z}} h\left(\frac{2k-1}{K}\right) \sum_{f\in H_{2k}(1)} \omega_f \sideset{}{^\flat}\sum_{d} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{\lambda_f(m)\lambda_f(n)\chi_d(m)\chi_d(n)}{\sqrt{mn}} V_k\left(\frac{m}{|d|}\right) V_k\left(\frac{n}{|d|}\right) \phi\left(\frac{(-1)^k d}{X}\right).$$

Now, applying the Petersson formula the previous display equals

$$4\sum_{k\in\mathbb{Z}} h\left(\frac{2k-1}{K}\right) \sideset{}{^\flat}\sum_{d} \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{\chi_d(mn)}{\sqrt{mn}} V_k\left(\frac{m}{|d|}\right) V_k\left(\frac{n}{|d|}\right) \phi\left(\frac{(-1)^k d}{X}\right)$$
$$\times \left(1_{m=n} + 2\pi i^{2k} \sum_{c=1}^{\infty} \frac{S(m,n;c)}{c} J_{2k-1}\left(\frac{4\pi\sqrt{mn}}{c}\right)\right).$$

Let us first focus on the diagonal term. We separate even and odd parts in the m-variable to see that these terms are given by

$$4\sum_{k\in\mathbb{Z}} h\left(\frac{2k-1}{K}\right) \sideset{}{^\flat}\sum_{d} \sum_{\substack{m=1\\(m,2)=1}}^{\infty} \sum_{j=0}^{\infty} \frac{\chi_d(m^2)}{2^j m} V_k\left(\frac{2^j m}{|d|}\right)^2 \phi\left(\frac{(-1)^k d}{X}\right)$$

using the observation that $\chi_d(4^j)=1$ for all $j\geq 0$ for odd d.

We again pick out the constraint d being squarefree by the identity (8.1) and so the diagonal contribution equals

$$4\sum_{k\in\mathbb{Z}} h\left(\frac{2k-1}{K}\right) \sum_{d\equiv 1\,(16)} \sum_{\substack{\alpha=1\\ \alpha^2|d}}^{\infty} \mu(\alpha) \sum_{\substack{m=1\\(m,2)=1}}^{\infty} \sum_{j=0}^{\infty} \frac{\chi_d(m^2)}{2^j m} V_k\left(\frac{2^j m}{|d|}\right)^2 \phi\left(\frac{(-1)^k d}{X}\right)$$

$$= 4\sum_{k\in\mathbb{Z}} h\left(\frac{2k-1}{K}\right) \sum_{\substack{\alpha=1\\(\alpha,2)=1}}^{\infty} \mu(\alpha) \sum_{\substack{m=1\\(m,2\alpha)=1}}^{\infty} \sum_{j=0}^{\infty} \frac{1}{2^j m} \sum_{d\equiv\overline{\alpha^2}\,(16)} \chi_d(m^2) V_k\left(\frac{2^j m}{\alpha^2|d|}\right)^2 \phi\left(\frac{(-1)^k d\alpha^2}{X}\right). \tag{9.1}$$

At this point we separate the α-sum into parts with $\alpha\leq Z$ and $\alpha>Z$, where the parameter $1\leq Z\leq X$ is chosen later. The terms with $\alpha>Z$ are easily estimated as in the computation of the first moment, but this time using the estimate

$$\sum_{k\in\mathbb{Z}} h\left(\frac{2k-1}{K}\right) \sum_{f\in\mathcal{B}_k} \omega_f L\left(\frac{1}{2}, f\otimes\chi_d\right)^2 \ll_\varepsilon K(|d|K)^\varepsilon, \tag{9.2}$$

which holds uniformly in $|d|\leq k^2$, to yield $\ll_\varepsilon (KX)^\varepsilon XK/Z$. The moment result (9.2) is certainly well-known, but being unable to find a suitable reference in the literature, we provide a quick rough sketch of the proof. Using an approximate functional equation the left-hand side is roughly

$$\sum_{k\in\mathbb{Z}} h\left(\frac{2k-1}{K}\right) \sum_{f\in\mathcal{B}_k} \sum_{m,n\ll(k|d|)^{1+\varepsilon}} \frac{\lambda_f(m)\lambda_f(n)\chi_d(m)\chi_d(n)}{\sqrt{mn}}.$$

Executing the sum over $\mathcal{B}_k$ using the Petersson formula yields a diagonal contribution, which is

$$\sum_{k\in\mathbb{Z}} h\left(\frac{2k-1}{K}\right) \sum_{m\ll(k|d|)^{1+\varepsilon}} \frac{\lambda_f(m)^2}{m} \ll_\varepsilon K(|d|K)^\varepsilon$$

using Deligne's bound for $\lambda_f(m)$, and an off-diagonal part

$$\sum_{k\in\mathbb{Z}} h\left(\frac{2k-1}{K}\right) i^{2k} \sum_{m,n\ll(k|d|)^{1+\varepsilon}} \sum_{c=1}^{\infty} \frac{S(m,n;c)}{c} J_{2k-1}\left(\frac{4\pi\sqrt{mn}}{c}\right).$$

This is in turn bounded by executing the k-sum and using the decay properties of $W_K^{(2)}$.

Returning to the main proof, we are left to deal with the terms corresponding to $\alpha\leq Z$. The strategy is again to execute the d-sum using Lemma 5.7. For this we define

$$F(\xi;x,y,z) := \phi\left(\frac{\xi}{x}\right) V_k\left(\frac{y}{\xi}\right) V_k\left(\frac{z}{\xi}\right).$$

In particular,

$$V_k\left(\frac{2^j m}{\alpha^2|d|}\right)^2 \phi\left(\frac{(-1)^k\alpha^2 d}{X}\right) = F\left(d; \frac{X}{(-1)^k\alpha^2}, \frac{2^j m}{(-1)^k\alpha^2}, \frac{2^j m}{(-1)^k\alpha^2}\right),$$

again as the support of ϕ is contained in the positive real numbers.

Thus the rest of the diagonal equals

$$\begin{aligned}
&4\sum_{k\in\mathbb{Z}} h\left(\frac{2k-1}{K}\right) \sum_{\substack{\alpha\le Z\\(\alpha,2)=1}} \mu(\alpha) \sum_{\substack{m=1\\(m,2\alpha)=1}}^{\infty} \sum_{j=0}^{\infty} \frac{1}{2^j m} \sum_{d\equiv\overline{\alpha^2}\,(16)} \chi_d(m^2) F\left(d; \frac{X}{(-1)^k\alpha^2}, \frac{2^j m}{(-1)^k\alpha^2}, \frac{2^j m}{(-1)^k\alpha^2}\right)\\
&= 4\sum_{k\in\mathbb{Z}} h\left(\frac{2k-1}{K}\right) \sum_{\substack{\alpha\le Z\\(\alpha,2)=1}} \mu(\alpha) \sum_{\substack{m=1\\(m,2\alpha)=1}}^{\infty} \sum_{j=0}^{\infty} \frac{1}{2^j m}\cdot\frac{1}{16m^2}\left(\frac{16}{m^2}\right)\\
&\qquad\times\sum_{\ell\in\mathbb{Z}}\left(\frac{\overline{\ell m^2\alpha^2}}{16}\right)\tau_\ell(m^2)\widehat{F}\left(\frac{\ell}{16m^2}; \frac{X}{(-1)^k\alpha^2}, \frac{2^j m}{(-1)^k\alpha^2}, \frac{2^j m}{(-1)^k\alpha^2}\right)\\
&= \frac{1}{4}\sum_{k\in\mathbb{Z}} h\left(\frac{2k-1}{K}\right) \sum_{\substack{\alpha\le Z\\(\alpha,2)=1}} \mu(\alpha) \sum_{\substack{m=1\\(m,2\alpha)=1}}^{\infty} \frac{1}{m^3}\sum_{j=0}^{\infty}\frac{1}{2^j}\sum_{\ell\in\mathbb{Z}}\tau_\ell(m^2)e\left(\frac{\ell\overline{\alpha^2}}{16}\right)\widehat{F}\left(\frac{\ell}{16m^2}; \frac{X}{(-1)^k\alpha^2}, \frac{2^j m}{(-1)^k\alpha^2}, \frac{2^j m}{(-1)^k\alpha^2}\right).
\end{aligned}$$

The main term arises when $\ell = 0$. This contribution is given by

$$\frac{1}{4}\sum_{k\in\mathbb{Z}} h\left(\frac{2k-1}{K}\right) \sum_{\substack{\alpha=1\\(\alpha,2)=1}}^{\infty} \mu(\alpha) \sum_{\substack{m=1\\(m,2\alpha)=1}}^{\infty} \frac{\varphi(m^2)}{m^3}\sum_{j=0}^{\infty}\frac{1}{2^j}\widehat{F}\left(0; \frac{X}{(-1)^k\alpha^2}, \frac{2^j m}{(-1)^k\alpha^2}, \frac{2^j m}{(-1)^k\alpha^2}\right) + O\left(\frac{XK^{1+\varepsilon}}{Z}\right)$$

by adding back the contribution of $\alpha > Z$ using the estimate

$$\widehat{F}(\lambda; x, y, z) \ll_A x\min\left\{\left(\frac{K^2x^2}{yz}\right)^A, \left(\frac{K^2}{\lambda|yz|}\right)^A\right\}. \tag{9.3}$$

But by definition

$$\widehat{F}\left(0; \frac{X}{(-1)^k\alpha^2}, \frac{2^j m}{(-1)^k\alpha^2}, \frac{2^j m}{(-1)^k\alpha^2}\right) = \int_{\mathbb{R}} V_k\left(\frac{2^j m}{(-1)^k\xi\alpha^2}\right)^2 \phi\left(\frac{(-1)^k\xi\alpha^2}{X}\right)\mathrm{d}\xi.$$

Using (5.4) the above may be written as

$$\begin{aligned}
&\widehat{F}\left(0; \frac{X}{(-1)^k\alpha^2}, \frac{2^j m}{(-1)^k\alpha^2}, \frac{2^j m}{(-1)^k\alpha^2}\right)\\
&= \frac{X}{\alpha^2}\int_{\mathbb{R}} \phi(\xi)V_k\left(\frac{2^j m}{X\xi}\right)^2\mathrm{d}\xi\\
&= \frac{X}{\alpha^2}\cdot\frac{1}{(2\pi i)^2}\int_{(\varepsilon)}\int_{(\varepsilon)} \frac{\Gamma(s_1+k)}{\Gamma(k)}\cdot\frac{\Gamma(s_2+k)}{\Gamma(k)}\left(\frac{X}{2\pi\cdot 2^j m}\right)^{s_1+s_2} e^{s_1^2+s_2^2}\left(\int_{\mathbb{R}}\phi(\xi)\xi^{s_1+s_2}\right)\frac{\mathrm{d}s_1\,\mathrm{d}s_2}{s_2 s_2}.
\end{aligned}$$

At this point we have written the part corresponding to $\ell = 0$ terms in the diagonal contribution into the form

$$\frac{X}{4}\sum_{k\in\mathbb{Z}} h\left(\frac{2k-1}{K}\right) \sum_{\substack{\alpha=1\\(\alpha,2)=1}}^{\infty} \frac{\mu(\alpha)}{\alpha^2}\frac{1}{(2\pi i)^2}\int_{(\varepsilon)}\int_{(\varepsilon)} \frac{\Gamma(s_1+k)}{\Gamma(k)}\cdot\frac{\Gamma(s_2+k)}{\Gamma(k)}\left(\frac{X}{2\pi}\right)^{s_1+s_2} e^{s_1^2+s_2^2}$$
$$\times\left(\sum_{\substack{m=1\\(m,2\alpha)=1}}^{\infty}\frac{\varphi(m^2)}{m^{3+s_1+s_2}}\right)\left(\sum_{j=0}^{\infty}\frac{1}{2^{j(1+s_1+s_2)}}\right)\left(\int_{\mathbb{R}}\phi(\xi)\xi^{s_1+s_2}\,\mathrm{d}\xi\right)\frac{\mathrm{d}s_1\,\mathrm{d}s_2}{s_1s_2}.$$

Note that for $\mathrm{Re}(s) > 0$ we have

$$\sum_{\substack{\alpha=1\\(\alpha,2)=1}}^{\infty}\frac{\mu(\alpha)}{\alpha^2}\sum_{(m,2\alpha)=1}\frac{\varphi(m^2)}{m^{3+s}} = \frac{\zeta(1+s)}{\zeta(2+s)}\sum_{\substack{\alpha=1\\(\alpha,2)=1}}^{\infty}\frac{\mu(\alpha)}{\alpha^2}\prod_{p|2\alpha}\frac{1-p^{-1-s}}{1-p^{-2-s}}$$
$$= \frac{\zeta(1+s)}{\zeta(2+s)}\cdot\frac{1-2^{-1-s}}{1-2^{-2-s}}\cdot\prod_{p\neq 2}\left(1-\frac{1}{p^2}\left(\frac{1-p^{-1-s}}{1-p^{-2-s}}\right)\right)$$

Let us denote

$$\mathcal{G}(s) := \prod_{p\neq 2}\left(1-\frac{1}{p^2}\left(\frac{1-p^{-1-s}}{1-p^{-2-s}}\right)\right) \qquad\text{and}\qquad \mathcal{H}(s) := \frac{1-2^{-1-s}}{1-2^{-2-s}}.$$

Estimating trivially it follows that the Euler product $\mathcal{G}(s)$ extends analytically to the domain $\mathrm{Re}(s) > -1$ and is bounded there.

Thus the expression for the $\ell = 0$ contribution can be written in the form

$$\frac{X}{4}\sum_{k\in\mathbb{Z}} h\left(\frac{2k-1}{K}\right)\frac{1}{(2\pi i)^2}\int_{(\varepsilon)}\int_{(\varepsilon)}\frac{\Gamma(s_1+k)}{\Gamma(k)}\cdot\frac{\Gamma(s_2+k)}{\Gamma(k)}\left(\frac{X}{2\pi}\right)^{s_1+s_2} e^{s_1^2+s_2^2}$$
$$\times\left(\int_{\mathbb{R}}\phi(\xi)\xi^{s_1+s_2}\,\mathrm{d}\xi\right)\left(\sum_{j=0}^{\infty}\frac{1}{2^{j(1+s_1+s_2)}}\right)\frac{\zeta(1+s_1+s_2)}{\zeta(2+s_1+s_2)}\mathcal{H}(s_1+s_2)\mathcal{G}(s_1+s_2)\frac{\mathrm{d}s_1\,\mathrm{d}s_2}{s_1s_2}. \tag{9.4}$$

We move the line of integration from $\mathrm{Re}(s_2) = \sigma$ to $\mathrm{Re}(s_2) = -1/2+\varepsilon$, crossing simple poles at $s_2 = 0$ and $s_2 = -s_1$. The integral on the new line is $O(X^{-1/2+\varepsilon})$. Next we shall study the contribution of these residues to (9.4) separately.

• Residue at $s_2 = 0$: This is given by

$$\frac{X}{4}\sum_{k\in\mathbb{Z}} h\left(\frac{2k-1}{K}\right)\frac{1}{2\pi i}\int_{(\varepsilon)}\frac{\Gamma(s_1+k)}{\Gamma(k)}\left(\frac{X}{2\pi}\right)^{s_1} e^{s_1^2}\left(\sum_{j=0}^{\infty}\frac{1}{2^{j(1+s_1)}}\right)\left(\int_{\mathbb{R}}\phi(\xi)\xi^{s_1}\,\mathrm{d}\xi\right)\frac{\zeta(1+s_1)}{\zeta(2+s_1)}\mathcal{H}(s_1)\mathcal{G}(s_1)\frac{\mathrm{d}s_1}{s_1}.$$

We shift the line of integration in the previous display to $\mathrm{Re}(s_1) = -1/2+\varepsilon$, crossing a double pole at $s_1 = 0$. The integral on the new line is $O(X^{-1/2+\varepsilon})$. By a straightforward computation using residue theorem shows that the previous display equals

$$\frac{X}{3\zeta(2)}\sum_{k\in\mathbb{Z}} h\left(\frac{2k-1}{K}\right)\int_{\mathbb{R}}\phi(\xi)\left(\mathcal{G}(0)\log\left(\frac{X\xi}{2\pi}\right)+\frac{\Gamma'(k)}{\Gamma(k)}\mathcal{G}(0)+\left(\frac{\log 2}{3}+\gamma\right)\cdot\mathcal{G}(0)-\frac{\zeta'(2)}{\zeta(2)}\mathcal{G}(0)+\mathcal{G}'(0)\right)\mathrm{d}\xi,$$

where γ is the Euler-Mascheroni constant.

• Residue at $s_1 = -s_2$: Likewise, using the series representation

$$\zeta(1+x) = \frac{1}{x}+\gamma+\cdots,$$

the term corresponding to this residue equals

$$-\frac{X}{3\zeta(2)}\cdot\mathcal{G}(0)\widehat{\phi}(0)\sum_{k\in\mathbb{Z}}h\left(\frac{2k-1}{K}\right)\cdot\frac{1}{2\pi i}\int\limits_{(\varepsilon)}\frac{\Gamma(s_1+k)}{\Gamma(k)}\cdot\frac{\Gamma(-s_1+k)}{\Gamma(k)}e^{s_1^2}\frac{\mathrm{d}s_1}{s_1^2}$$

Combining the above calculations we conclude that the the $\ell=0$ contribution of the diagonal terms equals

$$\frac{X}{3\zeta(2)}\sum_{k\in\mathbb{Z}}h\left(\frac{2k-1}{K}\right)\int\limits_{\mathbb{R}}\phi(\xi)\left(\mathcal{G}(0)\log\left(\frac{X\xi}{2\pi}\right)+\frac{\Gamma'(k)}{\Gamma(k)}\mathcal{G}(0)+\left(\frac{\log 2}{3}+\gamma\right)\cdot\mathcal{G}(0)-\frac{\zeta'(2)}{\zeta(2)}\mathcal{G}(0)+\mathcal{G}'(0)\right)\mathrm{d}\xi$$
$$-\frac{X}{3\zeta(2)}\cdot\mathcal{G}(0)\widehat{\phi}(0)\sum_{k\in\mathbb{Z}}h\left(\frac{2k-1}{K}\right)\cdot\frac{1}{2\pi i}\int\limits_{(\varepsilon)}\frac{\Gamma(s_1+k)}{\Gamma(k)}\cdot\frac{\Gamma(-s_1+k)}{\Gamma(k)}e^{s_1^2}\frac{\mathrm{d}s_1}{s_1^2}.$$

The first line can be simplified as follows. Using the simple observation that $\log k=\log(K\cdot(k-1)/K)+O(k^{-1})$ along with the asymptotic formula

$$\frac{\Gamma'(k)}{\Gamma(k)}=\log(k-1)+O\left(\frac{1}{k}\right),$$

and then evaluating the k-sum by Poisson summation, the above contribution may be written, up to an error $O(K^{\varepsilon}X)$, as

$$\frac{XK}{6\zeta(2)}\Bigg(\mathcal{G}(0)\widehat{h}(0)\widehat{\phi}(0)\log X+\mathcal{G}(0)\widehat{h}(0)\widehat{\phi}(0)\log K$$
$$+\mathcal{G}(0)\widehat{h}(0)\int\limits_{\mathbb{R}}\phi(\xi)\log\xi\,\mathrm{d}\xi+\mathcal{G}(0)\widehat{\phi}(0)\int\limits_{\mathbb{R}}h(\xi)\log\xi\,\mathrm{d}\xi+\widehat{\phi}(0)\widehat{h}(0)\cdot\mathcal{C}\Bigg)\tag{9.5}$$

where the constant $\mathcal{C}$ is given by

$$\mathcal{C}:=\left(\frac{\log 2}{3}+\gamma-\frac{\zeta'(2)}{\zeta(2)}-\log(2\pi)\right)\mathcal{G}(0)+\mathcal{G}'(0).\tag{9.6}$$

For the second term we observe that using the observation

$$\int\limits_{(\varepsilon)}\frac{\Gamma(s+k)\Gamma(-s+k)}{\Gamma(k)^2}e^{s^2}\frac{\mathrm{d}s}{s^2}=\int\limits_{(\varepsilon)}\frac{e^{s^2}}{s^2}\,\mathrm{d}s+O(K^{-1+\varepsilon})$$

(which follows from (5.6)) and treating the k-sum by Poisson summation, it is given by

$$-\frac{XK}{6\zeta(2)}\cdot\mathcal{G}(0)\widehat{\phi}(0)\widehat{h}(0)\cdot\frac{1}{2\pi i}\int\limits_{(\varepsilon)}\frac{e^{s^2}}{s^2}\,\mathrm{d}s+O\left(XK^{\varepsilon}\right).\tag{9.7}$$

It will turn out that the main part of this is cancelled by the off-diagonal.

Before proving that, observe that the contribution of the terms $\ell\neq 0$ of the diagonal is given by

$$\frac{1}{4}\sum_{k\in\mathbb{Z}}h\left(\frac{2k-1}{K}\right)\sum_{\substack{\alpha\leq Z\\(\alpha,2)=1}}\mu(\alpha)\sum_{\substack{m=1\\(m,2\alpha)=1}}^{\infty}\frac{1}{m^3}\sum_{j=0}^{\infty}\frac{1}{2^j}\sum_{\ell\neq 0}\tau_\ell(m^2)e\left(\frac{\ell\overline{\alpha^2}}{16}\right)\widehat{F}\left(\frac{\ell}{16m^2};\frac{X}{(-1)^k\alpha^2},\frac{2^jm}{(-1)^k\alpha^2},\frac{2^jm}{(-1)^k\alpha^2}\right)$$

By (9.3) the terms with $|\ell|\geq 16k^2\alpha^2(KX)^{\varepsilon}$ contribute negligible as the Fourier transform has a rapid decay. The smaller ℓ's can be treated exactly as the analogous terms in the second moment computation in the proof of Lemma 2.7. We write $\widehat{F}$ in terms of its Mellin transform, shift the line of integration to $\sigma=-1+\varepsilon$ and finally estimate everything trivially. This gives the total contribution $\ll_\varepsilon K^{1+\varepsilon}Z$. The optimal choice for Z is again $Z=X^{1/2}$.

We now proceed to showing that the part (9.7) of the diagonal is cancelled by the main contribution of the off-diagonal. The latter is given by

$$8\pi \sum_{k\in\mathbb{Z}} h\left(\frac{2k-1}{K}\right) i^{2k} \sum_{d}^{\flat}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{\chi_d(m)\chi_d(n)}{\sqrt{mn}} V_k\left(\frac{m}{|d|}\right) V_k\left(\frac{n}{|d|}\right) \phi\left(\frac{(-1)^k d}{X}\right) \times \sum_{c=1}^{\infty} \frac{S(m,n;c)}{c} J_{2k-1}\left(\frac{4\pi\sqrt{mn}}{c}\right).$$

In what follows we temporarily think d as fixed and postpone the summation over d to the last moment. We execute the k-sum using (5.12) to see that the off-diagonal equals

$$-2\sqrt{\pi}K \sum_{d}^{\flat}\phi\left(\frac{d}{X}\right)\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{\chi_d(m)\chi_d(n)}{m^{3/4}n^{3/4}} \times \operatorname{Im}\left(e^{-2\pi i/8}\sum_{c=1}^{\infty}\frac{S(m,n;c)}{\sqrt{c}}\cdot e\left(\frac{2\sqrt{mn}}{c}\right) W_K^{(2)}\left(\frac{m}{|d|},\frac{n}{|d|},\frac{K^2c}{8\pi\sqrt{mn}}\right)\right) + O\left(X^3/K^2\right), \tag{9.8}$$

where the error comes from estimating the error term in (5.12) trivially.

Splitting the m- and n-sums into congruence classes modulo $[c,d]$, these sums may be written as

$$\sum_{x\ ([c,d])}\sum_{w\ ([c,d])} S(x,w;c)\chi_d(x)\chi_d(w) \times \sum_{m\equiv x\ ([c,d])}\sum_{n\equiv w\ ([c,d])} m^{-3/4}n^{-3/4} e\left(\frac{2\sqrt{mn}}{c}\right) W_K^{(2)}\left(\frac{m}{|d|},\frac{n}{|d|},\frac{K^2c}{8\pi\sqrt{mn}}\right). \tag{9.9}$$

Applying Poisson summation to the m-sum gives

$$\frac{1}{[c,d]}\sum_{v} e\left(\frac{xv}{[c,d]}\right)\int_{\mathbb{R}} y^{-3/4} e\left(\frac{2\sqrt{yn}}{c}-\frac{yv}{[c,d]}\right) W_K^{(2)}\left(\frac{y}{|d|},\frac{n}{|d|},\frac{K^2c}{8\pi\sqrt{yn}}\right)\mathrm{d}y.$$

Similarly applying Poisson summation to the n-sum gives that after some computations our sum equals

$$\sum_{x\ ([c,d])}\sum_{w\ ([c,d])} S(x,w;c)\,\chi_d(x)\chi_d(w)\frac{1}{[c,d]^2}\sum_{v}\sum_{\eta} e\left(\frac{xv+w\eta}{[c,d]}\right)\cdot\mathcal{I},$$

where

$$\mathcal{I} := \int_{\mathbb{R}}\int_{\mathbb{R}} y^{-3/4}z^{-3/4} e\left(\frac{2\sqrt{yz}}{c}-\frac{yv}{[c,d]}-\frac{z\eta}{[c,d]}\right) W_K^{(2)}\left(\frac{y}{|d|},\frac{z}{|d|},\frac{K^2c}{8\pi\sqrt{yz}}\right)\mathrm{d}y\,\mathrm{d}z.$$

We first focus on the y-integral first. Let us denote

$$f(y) := \frac{2\sqrt{yz}}{c}-\frac{yv}{[c,d]}-\frac{z\eta}{[c,d]}.$$

An easy computation shows that the integral has a saddle-point at

$$y_0 := \frac{[c,d]^2 z}{c^2v^2}.$$

Similarly a straightforward computation gives

$$f(y_0) = z\left(\frac{\ell[c,d]}{c^2v}-\frac{\eta}{[c,d]}\right) \qquad \text{and} \qquad f''(y_0) = -\frac{1}{2}\frac{c^2v^3}{z[c,d]^3}.$$

Truncating the y-integral smoothly to $|y|\leq K^{1+\varepsilon}|d|$ with a negligible error using the rapid decay of $W_K^{(2)}$ and then applying Lemma 5.9 with the choices $V_1 = Q = K^{1+\varepsilon}|d|$, $X = (|d|K^{1+\varepsilon})^{-3/4}$, $Y = \sqrt{z}(K^{1+\varepsilon}|d|)^{1/2}/c$, and $V=(K^{1+\varepsilon}|d|)^{3/4}$ it follows that the expression (9.8) is, after some simplification, equal to

$$-2\sqrt{2\pi}K\sum_{d}{}^{\flat}\phi\left(\frac{d}{X}\right)\mathrm{Im}\Bigg(e^{-\pi i/2}\sum_{c=1}^{\infty}\frac{1}{[c,d]^2}\sum_{v}\sum_{\eta}\sum_{x\,([c,d])}\sum_{w\,([c,d])}\chi_d(x)\chi_d(w)$$

(9.10)

$$\times S(x,w;c)e\left(\frac{xv+w\eta}{[c,d]}\right)\int_{\mathbb{R}}z^{-1}e\left(z\left(\frac{[c,d]}{c^2v}-\frac{\eta}{[c,d]}\right)\right)W_K^{(2)}\left(\frac{z[c,d]^2}{v^2c^2|d|},\frac{z}{|d|},\frac{K^2c^2v}{8\pi z[c,d]}\right)\mathrm{d}z\Bigg)+O\left(X^{3/4}K^{1+\varepsilon}\right).$$

Note that the exponential phase in the last integral vanishes when $\eta v=[c,d]^2/c^2$. If this not the case, we may bound the integral using the first derivative test [17, Section 1.5.]. Indeed, note that in this case the absolute value of the phase function is $\geq 1/c^2v[c,d]$. Also by (5.9) the weight function $W^{(2)}$ is negligible unless

$$v\ll\frac{[c,d]|d|K^{\varepsilon}}{c^2K}$$

and the same holds for η. Note that as $|d|\ll X\ll\sqrt{K}$ this shows that the integral is negligible unless $v,\eta\leq K^{\varepsilon}/c$, which effectively truncates also the c-sum at K^{ε}. Now using the first derivative test and estimating everything else trivially using the Weil bound for Kloosterman sums shows that (9.10) is $\ll X^{3/4}K^{1+\varepsilon}$.

When $\eta v=[c,d]^2/c^2$ we use Lemma 7.1 to deduce that the main part of the off-diagonal is

$$-2\sqrt{2\pi}K\frac{\varphi(d)}{d}\mathrm{Im}\left(e^{-\pi i/2}\sum_{c=1}^{\infty}\int_{\mathbb{R}}z^{-1}W_K^{(2)}\left(\frac{z}{|d|},\frac{z}{|d|},\frac{K^2c|d|}{8\pi z}\right)\mathrm{d}z\right). \tag{9.11}$$

Making the change of variables $z\mapsto cz$, using the approximation (5.13) and Poisson summation we see that the integral above equals

$$\int_{\mathbb{R}}z^{-1}W\left(\frac{z}{K|d|},\frac{z}{K|d|},\frac{K^2c|d|}{8\pi z}\right)=\sum_{n=1}^{\infty}n^{-1}W\left(\frac{nc}{K|d|},\frac{nc}{K|d|},\frac{K^2|d|}{8\pi n}\right)$$

up to a small error.

By Mellin inversion we have, for any $A>2$,

$$\begin{aligned}&W\left(\frac{nc}{K|d|},\frac{nc}{K|d|},\frac{K^2|d|}{8\pi n}\right)\\&=\frac{1}{(2\pi i)^2}\int_{(A)}\int_{(A)}(2\pi)^{-x-y}e^{x^2+y^2}\left(\frac{nc}{K|d|}\right)^{-x}\left(\frac{nc}{K|d|}\right)^{-y}\hbar_{x+y}\left(\frac{K^2|d|}{8\pi n}\right)\frac{\mathrm{d}x\,\mathrm{d}y}{xy}\\&=\frac{1}{(2\pi i)^3}\int_{(A)}\int_{(A)}\int_{(1-\varepsilon)}(2\pi)^{-x-y}e^{x^2+y^2}\left(\frac{nc}{K|d|}\right)^{-x}\left(\frac{nc}{K|d|}\right)^{-y}\left(\frac{8\pi n}{K^2|d|}\right)^{z}\widetilde{\hbar}_{x+y}(z)\,\mathrm{d}z\,\frac{\mathrm{d}x\,\mathrm{d}y}{xy},\end{aligned}$$

where we have used (5.15) in the final step.

Substituting these into (9.11) leads us to consider

(9.12)

$$\frac{1}{(2\pi i)^3}\int_{(A)}\int_{(A)}\int_{(1-\varepsilon)}\zeta(1+x+y-z)\zeta(x+y)(2\pi)^{-x-y}e^{x^2+y^2}\left(\frac{1}{K|d|}\right)^{-x}\left(\frac{1}{K|d|}\right)^{-y}\left(\frac{8\pi}{K^2|d|}\right)^{z}\widetilde{\hbar}_{x+y}(z)\,\mathrm{d}z\,\frac{\mathrm{d}x\,\mathrm{d}y}{xy}.$$

Using the representation given in Lemma 5.5 for the Mellin transform of $\hbar_{x+y}$ and shifting the y-integral to the line $\sigma=\varepsilon$ we cross simple poles at $y=z-x$ with residue 1. Thus (9.12) takes the form

$$\frac{1}{(2\pi i)^2}\int_{(A)}\int_{(1-\varepsilon)}(2\pi K)^{-z}\zeta(z)\widetilde{\hbar}_z(z)e^{x^2+(z-x)^2}\frac{\mathrm{d}x\,\mathrm{d}z}{x(z-x)}.$$

Shifting the z-integral to the line $\sigma = -\varepsilon$ we encounter a simple pole at $z = 0$ coming from $\widetilde{h}_z(z)$ (this can be seen from the expression (5.14) which continues to hold for $\mathrm{Re}(s) > -1/2$) with residue

$$\int_0^\infty \frac{h(\sqrt{u})}{\sqrt{2\pi u}}\,\mathrm{d}u = \sqrt{\frac{2}{\pi}}\int_0^\infty h(u)\,\mathrm{d}u = \sqrt{\frac{2}{\pi}}\widehat{h}(0)$$

and so our integral equals

$$-\frac{\widehat{h}(0)}{\sqrt{2\pi}}\cdot\frac{1}{2\pi i}\int_{(A)}\frac{e^{x^2}}{x^2}\,\mathrm{d}x, \tag{9.13}$$

where we have also used the fact that $\zeta(0) = -1/2$. Now we are ready to perform the d-summation. We have shown that at this point the main contribution from the off-diagonal equals

$$-2\sqrt{2\pi}\left(-\frac{1}{\sqrt{2\pi}}\right)\widehat{h}(0)K\sideset{}{^\flat}\sum_{d}\frac{\varphi(d)}{d}\phi\left(\frac{d}{X}\right)\frac{1}{2\pi i}\int_{(A)}\frac{e^{x^2}}{x^2}\,\mathrm{d}x. \tag{9.14}$$

We now evaluate the sum sum over d, that is,

$$\sideset{}{^\flat}\sum_{d}\frac{\varphi(d)}{d}\phi\left(\frac{d}{X}\right).$$

Detecting the squarefree constraint using (8.1) and using Mellin inversion this sum is given by

$$\frac{1}{2\pi i}\int_{(A)}\widetilde{\phi}(s)X^s\left(\sum_{\substack{\alpha=1\\(\alpha,2)=1}}^{\infty}\mu(\alpha)\sum_{\substack{d\equiv 1\,(16)\\ \alpha^2\mid d}}\frac{\varphi(d)}{d^{1+s}}\right)\mathrm{d}s. \tag{9.15}$$

Observe that

$$\sum_{\substack{\alpha=1\\(\alpha,2)=1}}^{\infty}\mu(\alpha)\sum_{\substack{d\equiv 1\,(16)\\ \alpha^2\mid d}}\frac{\varphi(d)}{d^{1+s}} = \frac{1}{8}\sum_{\substack{\alpha=1\\(\alpha,2)=1}}^{\infty}\mu(\alpha)\sum_{\substack{d\\ \alpha^2\mid d}}\frac{\varphi(d)}{d^{1+s}}\sum_{\psi\,(16)}\psi(d),$$

where the inner sum is over Dirichlet characters modulo 16. It is easily seen that non-principal characters give a small contribution. Indeed, the contribution coming from any such character ψ is at most

$$\frac{1}{2\pi i}\int_{(A)}\widetilde{\phi}(s)X^s\left(\sum_{d=1}^{\infty}\frac{\varphi(d)\psi(d)}{d^{1+s}}\right)\mathrm{d}s = \frac{1}{2\pi i}\int_{(A)}\widetilde{\phi}(s)X^s\frac{L(s,\psi)}{L(s+1,\psi)}\,\mathrm{d}s,$$

which is $\ll_\varepsilon X^\varepsilon$ by shifting the contour of integration to the line $\sigma = \varepsilon > 0$.

For the principal character contribution we compute

$$
\begin{aligned}
&\sum_{\substack{\alpha=1\\(\alpha,2)=1}}^{\infty}\mu(\alpha)\sum_{\substack{(d,2)=1\\ \alpha^2|d}}\frac{\varphi(d)}{d^{1+s}}\\
&=\sum_{\substack{\alpha=1\\(\alpha,2)=1}}^{\infty}\frac{\mu(\alpha)}{\alpha^{2s}}\sum_{(d,2)=1}\frac{\varphi(d)}{d^{1+s}}\prod_{\substack{p|\alpha\\ p\nmid d}}\left(1-\frac{1}{p}\right)\\
&=\sum_{(d,2)=1}\frac{\varphi(d)}{d^{1+s}}\prod_{p|d}\left(1-\frac{1}{p^{2s}}\right)\prod_{p\nmid d}\left(1-\frac{1}{p^{2s}}\left(1-\frac{1}{p}\right)\right)\\
&=\prod_{p\neq 2}\left(1-\frac{1}{p^{2s}}\left(1-\frac{1}{p}\right)+\sum_{j=1}^{\infty}\frac{\varphi(p^j)}{p^{j(s+1)}}\left(1-\frac{1}{p^{2s}}\right)\right)\\
&=\prod_{p\neq 2}\left(1-\frac{1}{p^{2s}}\left(1-\frac{1}{p}\right)+\frac{1}{p^s}\left(1-\frac{1}{p^{2s}}\right)\left(1-\frac{1}{p}\right)\left(1-\frac{1}{p^s}\right)^{-1}\right)\\
&=\frac{\zeta(s)}{\zeta(1+s)}\prod_{p\neq 2}\left(1-\frac{1}{p^{2s}}\left(1-\frac{1}{p}\right)+\frac{1}{p^s}\left(1-\frac{1}{p^{2s}}\right)\left(1-\frac{1}{p}\right)\left(1-\frac{1}{p^s}\right)^{-1}\right)\frac{(1-p^{-s})}{(1-p^{-s-1})}\cdot\frac{1-2^{-s}}{1-2^{-s-1}}\\
&=\frac{\zeta(s)}{\zeta(1+s)}\prod_{p\neq 2}\left(1-\frac{1}{p^{s+1}}-\frac{1}{p^{2s}}\left(1-\frac{1}{p}\right)\right)\left(1-p^{-s-1}\right)^{-1}\cdot\frac{1-2^{-s}}{1-2^{-s-1}}.
\end{aligned}
$$

From this we see that the integrand has a simple pole at $s=1$ and so shifting the line of integration to the line $\mathrm{Re}(s)=\varepsilon$ in (9.15) we see that, up to a small error, the d-sum is

$$\frac{X}{12}\cdot\frac{\widetilde{\phi}(1)}{\zeta(2)}\prod_{p\neq 2}\left(1-\frac{1}{p^2}\left(\frac{1-p^{-1}}{1-p^{-2}}\right)\right)$$

and so, combining with (9.14), the main contribution from the off-diagonal equals

$$\frac{XK}{6}\widehat{h}(0)\cdot\frac{\widetilde{\phi}(1)}{\zeta(2)}\prod_{p\neq 2}\left(1-\frac{1}{p^2}\left(\frac{1-p^{-1}}{1-p^{-2}}\right)\right)\cdot\frac{1}{2\pi i}\int_{(\varepsilon)}\frac{e^{x^2}}{x^2}\,\mathrm{d}x=\frac{XK}{6\zeta(2)}\widehat{h}(0)\widehat{\phi}(0)\mathcal{G}(0)\cdot\frac{1}{2\pi i}\int_{(\varepsilon)}\frac{e^{x^2}}{x^2}\,\mathrm{d}x.$$

by noting that $\widetilde{\phi}(1)=\widehat{\phi}(0)$. This is precisely the inverse of (9.7), as claimed. □

10. Proof of Theorem 1.1

We first prove Theorem 1.1 assuming the truth of Propositions 2.5 and 2.6. Let $\alpha\in\{-\frac{1}{2},0\}$, $\varepsilon>0$ be any fixed small constant, and let K be large positive parameter. Recall that as $g(\alpha+iy)$ is real-valued for these values of α, Proposition 6.1 yields information on the zeroes inside the Siegel sets

$$\mathcal{F}_Y:=\{z\in\Gamma_0(4)\backslash\mathbb{H}:\mathrm{Im}(z)\geq Y\}$$

with $c_1'\sqrt{k\log k}\leq Y\leq c_2'k$ for some positive constants c_1' and c_2'. Let η be as in Proposition 6.1. It follows immediately from that result that if we can find numbers $\ell_1,\ell_2\in]c_1,c_2k/Y[$ so that

$$\sqrt{\alpha_g}c_g(\ell_1)e(\alpha\ell_1)<-k^{-\delta}<k^{-\delta}<\sqrt{\alpha_g}c_g(\ell_2)e(\alpha\ell_2) \tag{10.1}$$

for some $\delta<1/2+\eta$, then $g(z)$ has a zero $\alpha+iy$ between y_{ℓ_1} and y_{ℓ_2}. Observe that

$$e(\alpha\ell)=\begin{cases}1 & \text{if } \alpha=0\\ (-1)^{\ell} & \text{if } \alpha=-1/2\end{cases} \tag{10.2}$$

Hence, in order to find real zeroes on the line $\mathrm{Re}(s)=0$ it suffices to detect sign changes among the Fourier coefficients whereas on the line $\mathrm{Re}(s)=-1/2$ one needs a sign change along odd arguments. As we restrict to odd fundamental discriminants for which $(-1)^k d > 0$, we automatically obtain real zeroes on both of the individual geodesic segments $\mathrm{Re}(s) = -1/2$ and $\mathrm{Re}(s) = 0$.

Remember that in order to detect sign changes along the sequence $d \equiv 1\,(16)$ with d squarefree and $(-1)^k d > 0$ on the short interval $[x, x+H]$ it suffices to have

$$\left| \sideset{}{^\flat}\sum_{x \le (-1)^k d \le x+H} \sqrt{\alpha_g} c_g(|d|) \right| < \sideset{}{^\flat}\sum_{x \le (-1)^k d \le x+H} \sqrt{\alpha_g} |c_g(|d|)|.$$

It follows easily from Propositions 2.5 and 2.6 that for $\gg K^2/(\log K)^{3/2}$ of the forms $g \in \mathcal{S}_K$ we have that

$$\left| \sideset{}{^\flat}\sum_{x \le (-1)^k d \le x+H} \sqrt{\alpha_g} |c_g(|d|)| \pm \sideset{}{^\flat}\sum_{x \le (-1)^k d \le x+H} \sqrt{\alpha_g} c_g(|d|) \right| \gg \frac{H}{\sqrt{k} \log X}$$

holds for $\gg X/(\log X)^{5/2}$ of the numbers $x \sim X$. We note that the contribution coming from the summands with $|\sqrt{\alpha_g} c_g(|d|)| \le k^{-\delta}$ is trivially bounded by

$$\ll \sum_{x \le (-1)^k d \le x+H} |\sqrt{\alpha_g} c_g(|d|)| \le 2HK^{-\delta}$$

and so we conclude, choosing $\delta = 1/2 + \eta/2$ for concreteness so that $\delta > 1/2$, that for the same proportion of $g \in \mathcal{S}_K$ and $x \sim X$ it holds that

$$\left| \sideset{}{^\flat}\sum_{\substack{x \le (-1)^k d \le x+H \\ |\sqrt{\alpha_g} c_g(|d|)| > k^{-\delta}}} \sqrt{\alpha_g} |c_g(|d|)| \pm \sideset{}{^\flat}\sum_{\substack{x \le (-1)^k d \le x+H \\ |\sqrt{\alpha_g} c_g(|d|)| > k^{-\delta}}} \sqrt{\alpha_g} c_g(|d|) \right| \gg \frac{H}{\sqrt{k} \log X}$$

in the view that the parameter H is chosen to be a fixed power of $\log K$. This leads to the promised number of real zeroes by the discussion above and in the introduction.

Next we shall show how the above two propositions just applied follow from lemmas 2.7 and 2.8. We start by proving the first key proposition.

10.1. **Proof of Proposition 2.5.** Recall that

$$\mathcal{S}_K = \bigcup_{k \sim K} B^+_{k+\frac{1}{2}}.$$

Denote

$$S_{1,g}(x; H) := \left| \sideset{}{^\flat}\sum_{x \le (-1)^k d \le x+H} \sqrt{\alpha_g} c_g(|d|) \right|$$

and

$$\mathcal{T}_{1,g}(X; H) := \# \left\{ x \sim X : \; S_{1,g}(x; H) \ge \sqrt{H} \cdot k^{-1/2} (\log K)^3 \right\}.$$

With this the quantity we need to bound is by Markov's inequality

$$\begin{aligned}
&\sum_{k\sim K} \sum_{\substack{g\in B^+_{k+\frac12}\\ |\mathcal{T}_{1,g}(X;H)|\geq X/(\log X)^3}} 1\\
&\leq \frac{(\log X)^3}{X}\sum_{k\in\mathbb{Z}} h\left(\frac{2k-1}{K}\right)\sum_{g\in B^+_{k+\frac12}} |\mathcal{T}_{1,g}(X;H)|\\
&\leq \frac{(\log X)^3}{X}\sum_{k\in\mathbb{Z}} h\left(\frac{2k-1}{K}\right)\sum_{g\in B^+_{k+\frac12}} \sum_{\substack{x\sim X\\ |S_{1,g}(x;H)|\geq \sqrt{H}k^{-1/2}(\log K)^3}} 1\\
&\ll \frac{K(\log X)^3}{H(\log K)^6 X}\sum_{k\in\mathbb{Z}} h\left(\frac{2k-1}{K}\right)\sum_{g\in B^+_{k+\frac12}} \alpha_g \sum_{x\sim X}\left|\sideset{}{^\flat}\sum_{x\leq (-1)^k d\leq x+H} c_g(|d|)\right|^2. \qquad (10.3)
\end{aligned}$$

By opening the absolute square the inner double sum can be rearranged into

$$\sideset{}{^\flat}\sum_{(-1)^k d\sim X}\sum_{|h|\leq H}(H-|h|)c_g(|d|)c_g(|d|+h).$$

Let us first focus on the diagonal $h=0$. In this case the total contribution to (10.3) is given by

$$\ll \frac{K(\log X)^3}{(\log K)^6 X}\sum_{k\in\mathbb{Z}} h\left(\frac{2k-1}{K}\right)\sum_{g\in B^+_{k+\frac12}} \alpha_g \sideset{}{^\flat}\sum_{(-1)^k d\sim X}|c_g(|d|)|^2.$$

The inner sum can be recasted in terms of central L-values by the Waldspurger formula. By Lemma 2.7 this is bounded by

$$\ll K^2\frac{(\log X)^3}{(\log K)^6}\ll \frac{K^2}{(\log K)^3},$$

as desired.

Let us then focus on the off-diagonal corresponding to the terms with $h\neq 0$. Notice that for $h\equiv 0,(-1)^{k+1}\,(4)$ we may apply Lemma 5.4 to see that

$$\sum_{g\in B_{k+\frac12}}\alpha_g c_g(|d|)c_g(|d|+h)=\sum_{4|c}\frac{K^+_\kappa(|d|,|d|+h;c)}{c}\cdot i^{2k}J_{2k-\frac32}\left(\frac{4\pi\sqrt{|d|(|d|+h)}}{c}\right),$$

and otherwise the sum on the left-hand side vanishes. Thus the off-diagonal part of our sum contributes

$$\begin{aligned}
&\ll \frac{K(\log X)^3}{H(\log K)^6 X}\sum_{k\in\mathbb{Z}} h\left(\frac{2k-1}{K}\right)\sum_{g\in B^+_{k+\frac12}}\alpha_g \sideset{}{^\flat}\sum_{(-1)^k d\sim X}\sum_{\substack{|h|\leq H\\ h\neq 0}}(H-|h|)c_g(|d|)c_g(|d|+h)\\
&\ll \frac{HK(\log X)^3}{(\log K)^6 X}\max_{0<|h|\leq H}\left|\sum_{k\in\mathbb{Z}} h\left(\frac{2k-1}{K}\right)\sum_{(-1)^k|d|\sim X}\sum_{4|c}\frac{K^+_\kappa(|d|,|d|+h;c)}{c}\cdot i^{2k}J_{2k-\frac32}\left(\frac{4\pi\sqrt{|d|(|d|+h)}}{c}\right)\right|.
\end{aligned}$$

As we are going to choose H to be a small power of $\log K$, we have $\sqrt{d(d+h)}\asymp X$. Moreover, $X\ll\sqrt{K}$ and so we may use the uniform estimate (5.18) and summing k-, d-, and c-sums trivially using (5.8), to see that off-diagonal terms contribute less than the diagonal. This completes the proof. □

10.2. **Proof of Proposition 2.6.** We start by deriving a lower bound for the weighted sum of the terms $|c_g(|d|)|$ on average over the forms $g \in \mathcal{S}_K$. Applying Hölder's inequality we have

$$\begin{aligned}
&\sum_{k\in\mathbb{Z}} h\left(\frac{2k-1}{K}\right) \sum_{g\in B^+_{k+\frac{1}{2}}} \alpha_g \sideset{}{^\flat}\sum_{(-1)^k d\sim X} |c_g(|d|)|^2 \\
&\le \left(\sum_{k\in\mathbb{Z}} h\left(\frac{2k-1}{K}\right) \sum_{g\in B^+_{k+\frac{1}{2}}} \sideset{}{^\flat}\sum_{(-1)^k d\sim X} \alpha_g^{1/2}\omega_g^{1/2}|c_g(|d|)|\right)^{2/3} \\
&\quad\times \left(\sum_{k\in\mathbb{Z})} h\left(\frac{2k-1}{K}\right) \sum_{g\in B^+_{k+\frac{1}{2}}} \sideset{}{^\flat}\sum_{(-1)^k d\sim X} \alpha_g^{2}\omega_g^{-1}|c_g(|d|)|^4\right)^{1/3}.
\end{aligned}$$

Using Lemmas 2.7 and 2.8 for the second and fourth moments of the Fourier coefficients respectively we obtain

$$\sum_{k\in\mathbb{Z}} h\left(\frac{2k-1}{K}\right) \sum_{g\in B^+_{k+\frac{1}{2}}} \sideset{}{^\flat}\sum_{(-1)^k d\sim X} \alpha_g^{1/2}\omega_g^{1/2}|c_g(|d|)| \gg \frac{XK}{(\log XK)^{1/2}}.$$

On the other hand, we also have

$$\begin{aligned}
&\sum_{k\in\mathbb{Z}} h\left(\frac{2k-1}{K}\right) \sum_{g\in B^+_{k+\frac{1}{2}}} \sideset{}{^\flat}\sum_{\substack{(-1)^k d\sim X\\ L(1/2,f\otimes\chi_d)>(\log XK)^2}} \alpha_g^{1/2}\omega_g^{1/2}|c_g(|d|)| \\
&\le (\log XK)^{-1} \sum_{k\in\mathbb{Z}} h\left(\frac{2k-1}{K}\right) \sum_{g\in B^+_{k+\frac{1}{2}}} \sideset{}{^\flat}\sum_{\substack{(-1)^k d\sim X\\ L(1/2,f\otimes\chi_d)>(\log XK)^2}} \alpha_g|c_g(|d|)|^2 \\
&\ll \frac{XK}{\log XK}
\end{aligned}$$

from which we conclude the lower bound

$$\sum_{k\in\mathbb{Z}} h\left(\frac{2k-1}{K}\right) \sum_{g\in B^+_{k+\frac{1}{2}}} \sideset{}{^\flat}\sum_{\substack{(-1)^k d\sim X\\ L(1/2,f\otimes\chi_d)\le(\log XK)^2}} \alpha_g^{1/2}\omega_g^{1/2}|c_g(|d|)| \gg \frac{KX}{(\log XK)^{1/2}}.$$

Let us now define the set

$$\mathcal{V}_g := \left\{ x\sim X : \sideset{}{^\flat}\sum_{\substack{x\le(-1)^k d\le x+H\\ L(1/2,f\otimes\chi_d)\le(\log XK)^2}} \sqrt{\alpha_g}|c_g(|d|)| \ge \frac{H}{k^{1/2}\log X} \right\}.$$

From the work above it follows that

$$\begin{aligned}\frac{KX}{(\log XK)^{1/2}} &\ll \sum_{k\in\mathbb{Z}} h\left(\frac{2k-1}{K}\right) \sum_{g\in B^+_{k+\frac12}} \sideset{}{^\flat}\sum_{\substack{(-1)^k d\sim X\\ L(1/2,f\otimes\chi_d)\le(\log XK)^2}} \alpha_g^{1/2}\omega_g^{1/2}|c_g(|d|)| \\ &\le \frac{1}{H}\sum_{k\in\mathbb{Z}} h\left(\frac{2k-1}{K}\right) \sum_{g\in B^+_{k+\frac12}} \sum_{x\sim X} \sideset{}{^\flat}\sum_{\substack{x\le(-1)^k d\le x+H\\ L(1/2,f\otimes\chi_d)\le(\log XK)^2}} \alpha_g^{1/2}\omega_g^{1/2}|c_g(|d|)| \\ &= \frac{1}{H}\sum_{k\in\mathbb{Z}} h\left(\frac{2k-1}{K}\right) \sum_{g\in B^+_{k+\frac12}} \left(\sum_{x\in\mathcal{V}_g}+\sum_{x\notin\mathcal{V}_g}\right) \sideset{}{^\flat}\sum_{\substack{x\le(-1)^k d\le x+H\\ L(1/2,f\otimes\chi_d)\le(\log XK)^2}} \alpha_g^{1/2}\omega_g^{1/2}|c_g(|d|)| \\ &\le \frac{1}{H}\sum_{k\in\mathbb{Z}} h\left(\frac{2k-1}{K}\right) \sum_{g\in B^+_{k+\frac12}} \left(\omega_g(\log XK)H\,|\mathcal{V}_g| + \omega_g^{1/2}\frac{XH}{k^{1/2}\log X}\right),\end{aligned}$$

where we have used the relation $\omega_g^{1/2}\alpha_g^{1/2}|c_g(|d|)| = \omega_g\sqrt{L(1/2,f\otimes\chi_d)}$ in the last estimate. Using an easy estimate $\sum_{g\in B^+_{k+\frac12}}\omega_g^{1/2} \ll \sqrt{k}$ (which follows from the Cauchy-Schwarz inequality and (2.6)) we conclude that

$$\sum_{k\in\mathbb{Z}} h\left(\frac{2k-1}{K}\right)\sum_{g\in B^+_{k+\frac12}} \omega_f\,|\mathcal{V}_g| \gg \frac{KX}{(\log XK)^{3/2}}.$$

We can remove the harmonic averaging using [2, Lemma 8.9.] to conclude that

$$\sum_{k\in\mathbb{Z}} h\left(\frac{2k-1}{K}\right)\sum_{g\in B^+_{k+\frac12}} |\mathcal{V}_g| \gg \frac{K^2X}{(\log XK)^{3/2}}. \tag{10.4}$$

Let us introduce the set

$$\mathcal{U} := \left\{g\in\mathcal{S}_K : |\mathcal{V}_g| > \frac{X}{(\log X)^{5/2}}\right\}.$$

Now from (10.4) we deduce that

$$\begin{aligned}\frac{XK^2}{(\log XK)^{3/2}} &\ll \sum_{g\in\mathcal{U}}|\mathcal{V}_g| + \sum_{g\in\mathcal{S}_K\setminus\mathcal{U}}|\mathcal{V}_g| \\ &\ll |\mathcal{U}|X + K^2\cdot\frac{X}{(\log X)^{5/2}}\end{aligned}$$

from which we infer the lower bound

$$|\mathcal{U}| \gg \frac{K^2}{(\log XK)^{3/2}}.$$

Hence we have shown that for $\gg K^2/(\log K)^{3/2}$ of the forms $g \in \mathcal{S}_K$ we have

$$\begin{aligned}
&\# \left\{ x \sim X \ : \ \sideset{}{^\flat}\sum_{x \leq (-1)^k d \leq x+H} \sqrt{\alpha_g} |c_g(|d|)| \geq \frac{H}{k^{1/2} \log X} \right\} \\
&\geq \# \left\{ x \sim X \ : \ \sideset{}{^\flat}\sum_{\substack{x \leq (-1)^k d \leq x+H \\ L(1/2, f \otimes \chi_d) \leq (\log XK)^2}} \sqrt{\alpha_g} |c_g(|d|)| \geq \frac{H}{k^{1/2} \log X} \right\} \\
&> \frac{X}{(\log X)^{5/2}},
\end{aligned}$$

which is what we wanted to prove. □

11. Proof of Theorem 1.3

First we explain how to use Lemmas 2.2 and 2.3 to deduce Proposition 2.1. Recall the definition of the mollifier $M_{f,d}$ from the introduction. Using the Cauchy-Schwarz inequality we have

$$\begin{aligned}
&\left(\sum_{k \in \mathbb{Z}} h\left(\frac{2k-1}{K} \right) \sum_{f \in \mathcal{B}_k} \omega_f L\left(\frac{1}{2}, f \otimes \chi_d \right) \mathcal{M}_{f,d} \right)^2 \\
&\leq \left(\sum_{k \sim K} \sum_{\substack{f \in \mathcal{B}_k \\ L\left(\frac{1}{2}, f \otimes \chi_d\right) \neq 0}} \omega_f \right) \left(\sum_{k \in \mathbb{Z}} h\left(\frac{2k-1}{K} \right) \sum_{f \in \mathcal{B}_k} \omega_f L\left(\frac{1}{2}, f \otimes \chi_d \right)^2 \mathcal{M}_{f,d}^2 \right)
\end{aligned}$$

from which we conclude using Lemmas 2.2 and 2.3 that

$$\sum_{k \sim K} \sum_{\substack{f \in \mathcal{B}_k \\ L\left(\frac{1}{2}, f \otimes \chi_d\right) \neq 0}} \omega_f \geq \left(\frac{1}{4} - \varepsilon \right) K \tag{11.1}$$

for all fundamental discriminants $|d| \leq k^\delta$ (with some $\delta > 0$) showing that for such d it holds that $c_g(|d|) \neq 0$ for a proportion $\geq 1/2 - \varepsilon$ of such forms g as discussed in the introduction. The harmonic weights ω_f may be removed again by using [2, Lemma 8.9.] (this is done already in [19]) to get the same proportion of non-vanishing for the natural average. This gives Proposition 2.1.

11.1. **Sign changes of Fourier coefficients along squares.** The final thing to do is to apply Proposition 2.1 to complete the proof. The idea is to connect short sums of the Fourier coefficients to long sums using the arguments of [38]. Fix any odd fundamental discriminant d. Given $g \in S_K$ we define the function $h_g : \mathbb{N} \longrightarrow \{-1, 0, 1\}$ by

$$h_g(m) := \begin{cases} 1 & \text{if } c_g(|d|) c_g(|d| m^2) > 0 \\ 0 & \text{if } c_g(|d|) c_g(|d| m^2) = 0 \\ -1 & \text{if } c_g(|d|) c_g(|d| m^2) < 0 \end{cases}$$

This is multiplicative, which follows from the fact that $m \mapsto c_g(|d| m^2)$ satisfy the multiplicative property $c_g(|d|) c_g(|d| m^2 n^2) = c_g(|d| m^2) c_g(|d| n^2)$ (when $(m,n) = 1$) [45, (1.18)] for any fixed fundamental discriminant d. First we note that for a proportion $1/2 - \varepsilon$ of forms $g \in \mathcal{S}_K$ the function h_g is not identically zero. Indeed, recall that for $g \in S^+_{k+\frac{1}{2}}(4)$ we have

$$c_g(|d| m^2) = c_g(|d|) \prod_{p | m} \left(\lambda_f(p) - \frac{\chi_d(p)}{\sqrt{p}} \right)$$

for $(-1)^k d > 0$. Now the assertion follows from Proposition 2.1 and the Sato-Tate law. For such a fixed form g, let p^ν be the smallest natural number for which $h_g(p^\nu) = -1$ (such an integer is necessarily a prime power). Write now $X = \sqrt{K/Y}$. Then using the fundamental lemma of the sieve we have

$$\sum_{m \le X} (|h_g(m)| - h_g(m)) \ge \sum_{\substack{m \le X/p^\nu \\ p \nmid m}} (|h_g(m)| - h_g(m) + |h_g(p^\nu m)| - h_g(p^\nu m))$$
$$= 2 \sum_{\substack{m \le X/p^\nu \\ p \nmid m}} |h_g(m)| \gg \frac{X}{p^\nu}.$$

An analogous argument shows that

$$\sum_{m \le X} (|h_g(m)| + h_g(m)) \gg \frac{X}{p^\nu}.$$

Now applying [38, Theorem 3] we conclude that for any $H \longrightarrow \infty$ along with $X \longrightarrow \infty$ it holds that

$$\sum_{x \le m \le x+H} (|h_g(m)| \pm h_g(m)) \gg \frac{H}{p^\nu}$$

for almost all $x \sim X$.

On the other hand, by Lemma 5.10 we know that for almost all forms $g \in \mathcal{S}_K$ we have $p^\nu \ll \log K$. Now combining the previous observations we get that for a proportion $1/2 - \varepsilon$ of forms $g \in \mathcal{S}_K$ the inequality

$$\sum_{x \le m \le x+H} (|h_g(m)| \pm h_g(m)) \gg \frac{H}{\log K}$$

holds for almost all $x \sim X$. This shows that for the forms g as above, $h_g(m)$ has a sign change in almost all short intervals of length, say $H = \log K$, implying that for a fixed d the sequence $c_g(|d|m^2)$ has $\gg X/\log K = (K/Y)^{1/2}(\log K)^{-1}$ sign changes, as desired. □

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